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**Johnson's determinantal identity for contiguous minors of Toeplitz matrices, with an accretive extension.** (English) Zbl 08284190

J. Algebra 711, 14-25 (2027).

For an $n \times n$ real matrix $A$, let $A_r(i,j)$ denote the $r \times r$ contiguous submatrix of $A$ whose top-left entry is $a_{ij}$, and let $J_n$ be the $n \times n$ all-ones matrix.

At a workshop on combinatorics, linear algebra and graph theory held in Tehran in 2003, Charles R. Johnson conjectured that every real Toeplitz matrix $A$ with $A + A^{\intercal} = 2J_n$, $n \geq 2$, satisfies

$$\det A_{n-1}(1,2) + \det A_{n-1}(2,1) = 2 \det A_{n-1}(1,1).$$

The concise paper under review proves this conjecture.

The proof writes $A = J_n + B$ with $B$ skew-symmetric Toeplitz and uses a rank-one expansion of $\det(X + tJ)$, together with elementary properties of the adjugate of a skew-symmetric matrix, to reduce the conjecture to an equality between two polynomial expressions in the superdiagonal entries $b_1, \ldots, b_{n-1}$ of $B$. The Desnanot-Jacobi identity shows that the squares of these two polynomials coincide, so the polynomials themselves agree up to a sign, which is determined by a suitable specialization of the parameters.

The second part concerns the arithmetic-geometric mean identity of *M. Bayat* and *H. Teimoori* [Linear Algebra Appl. 435, No. 11, 2936–2941 (2011; Zbl 1232.15005)], which asserts that

$$\sqrt{\det A_{n-1}(1,1)\, \det A_{n-1}(2,2)} = \left| \frac{\det A_{n-1}(1,2) + \det A_{n-1}(2,1)}{2} \right|$$

whenever $A + A^{\intercal} = \alpha J_n$ for some $\alpha \in \mathbb{R}$. By a diagonal scaling followed by a perturbation argument, the author extends this to all $A$ with $A + A^{\intercal} = \alpha\, ww^{\intercal}$, $\alpha \in \mathbb{R}$, $w \in \mathbb{R}^n \setminus \{0\}$. He then shows that if the symmetric part of $A$ is positive semidefinite, then

$$\sqrt{\det A_{n-1}(1,1)\, \det A_{n-1}(2,2)} \geq \left| \frac{\det A_{n-1}(1,2) + \det A_{n-1}(2,1)}{2} \right|.$$

The key observation is that $\operatorname{adj}(A)$ also has positive semidefinite symmetric part. When the symmetric part of $A$ is positive definite, this follows from $\operatorname{adj}(A) = (\det A)A^{-1}$, the positivity of $\det A$, and the fact that $A^{-1}$ has positive definite symmetric part. The general case is obtained by approximation with $A + \varepsilon I$. The desired inequality is then a consequence of the nonnegativity of the $2 \times 2$ principal minor of the symmetric part of $\operatorname{adj}(A)$ indexed by $\{1, n\}$, since the corresponding entries of $\operatorname{adj}(A)$ are, up to sign, the four contiguous minors. A $2 \times 2$ example shows that the corresponding inequality, with an absolute value around the product under the square root, fails even for complex matrices with positive definite Hermitian part.

Reviewer: Frédéric Morneau-Guérin (Québec)

**MSC:**

15B05 Toeplitz, Cauchy, and related matrices
15A15 Determinants, permanents, traces, other special matrix functions
15A42 Inequalities involving eigenvalues and eigenvectors

**Keywords:**

Toeplitz matrix; contiguous minors; skew-symmetric matrix; Desnanot-Jacobi identity; rank-one update; accretive matrix; adjugate

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