Benchmark, Relative Return, and Asset Pricing: The Full Version

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This unpublished working paper represents the full version of the article *Benchmark, Relative Return, and Asset Pricing*.* The present version provides additional details regarding the derivation of the model, including an explicit regularisation argument. See also the appendix A.

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Abstract

In this working paper, we develop a simple asset pricing model using the relative return to a benchmark. The model makes no assumption on free-risk securities, equilibrium conditions, utility functions, diffusion processes, probability distributions, or return generating processes. Our main result indicates that the asset's expected return is equal to the expected return of the lowest-risk asset, plus a risk premium directly proportional to the covariance between the asset's excess return and the benchmark factor. This suggests that an asset pricing model can be built without restrictive assumptions. This also suggests that the classic CAPM can be viewed as a special case of our benchmark model.

Keywords: Asset pricing, Benchmark, Relative return

JEL Classification Codes: G11, G12

I. Introduction

There are many extension models following the capital asset pricing model (CAPM) of Sharpe (1964) and Lintner (1965).

In this note, we develop a simple asset pricing model using the relative return to a benchmark.

The use of a benchmark portfolio for the valuation of assets represents a growing field of research in the *mathematical finance literature*. This *benchmark approach* was developed in a continuous-time framework by Platen (2006), and Platen and Heath (2006). In this set-up, the CAPM follows without the use of expected utility functions, equilibrium assumptions, or arbitrage restrictions. Along this line, many models confirm the popularity

of the benchmark approach. Examples include Karatzas and Kardaras (2007), Platen and Rendek (2012), Du and Platen (2016), and Curchiero et al. (2019).

However, the above *benchmark approach* requires the following assumptions: (1) the economy is limited to special markets with continuous security prices, (2) the trading uncertainty is expressed by independent standard Wiener processes, and (3) the savings account is in net zero supply. Our model makes none of these restrictive assumptions. Moreover, our model makes no restriction on free-risk securities, equilibrium (and arbitrage) conditions, utility functions, diffusion processes, probability distributions, or return generating processes. It considers a standard discrete-time framework, and its central assumption simply supposes that investors estimate the expected relative return to a benchmark.

Our main result indicates that the asset's expected return is equal to the expected return of the lowest-risk asset (or portfolio), plus a risk premium directly proportional to the covariance between the asset's excess return and the benchmark factor. This suggests that the classic CAPM can be viewed as a special case of our benchmark model.

In this sense, the main contributions of this paper can be summarized as follows. First, this paper demonstrates that we can characterize the risk-return relationship without restrictive assumptions (or under very weak assumptions). Second, it allows us to obtain similar results to the continuous benchmark approach, in an easier mathematical framework. Moreover, it allows us to avoid three restrictive assumptions (mentioned above) required by the benchmark approach (for asset valuation).

In Section 2, we expose our assumptions, build our framework, and derive our main results. In Section 3, we conclude. In appendix A, we discuss the theoretical robustness of the model.

II. The model

In this section, we develop our extension model. We first present the unrestrictive assumptions. Next, we derive the risk-return relationship using the relative return to a benchmark. Then we show that the classic CAPM can be viewed as a special case of our model. Thereafter, we focus on the lowest-risk asset and benchmark index.

The assumptions

By definition, the link between the *Absolute Return*, *Benchmark Return*, and *Relative Return*, is given by

$$1 + \textit{Absolute Return} = (1 + \textit{Benchmark Return}) \times (1 + \textit{Relative Return}).$$

Thus, we have

$$1 + \textit{Relative Return} = (1 + \textit{Absolute Return}) / (1 + \textit{Benchmark Return}).$$

Our model is based on the definitions above, and on the following (unrestrictive) assumptions:

A1 In the economy, there are many different assets and distinct investors;

A2 Investors prefer more rather than less, and are risk averse;

A3 For each asset, investors estimate the expected relative return to the benchmark.

Assumptions A1 and A2 are very easy to accept. Assumption A3 is also easy to accept, if we recognize the importance of relative motion in modern physics, and admit the usefulness of relative returns (or active returns) in portfolio management. In fact, it is reasonable to believe that a well-informed investor (or portfolio manager) will try to compare the expected returns of an investment to a reference point. In the next subsection, we will demonstrate that these three assumptions are sufficient to characterize the relationship between the risk of an asset and its expected return.

The risk-return relationship

Given the available information at time t, we suppose that investor k ($k = 1,2, \dots, K$) estimates the asset's expected relative return to the benchmark, as is shown below:

$$1 + \mu_{itk} = E_{tk}\left[(1 + \tilde{R}^k_{i,t+1})/(1 + \tilde{R}^k_{b,t+1})\right], \quad (1)$$

where $\tilde{R}^k_{i,t+1}$ is the return of asset i ($i = 1, 2, \dots, N$) at time $t + 1$, for investor k, $\tilde{R}^k_{b,t+1}$ is the return of the benchmark portfolio b, at time $t + 1$, for investor k, and μ_{itk} is the corresponding expected relative return.[1] If we put the ($K\ x\ N$) values of μ_{itk} in order from lowest to highest, we can identify the associated median ($\bar{\mu}_t$). We define the *median investor* as the one who estimates this median value. Thus, for the median investor, denoted by the letter d, we have

$$1 + \mu_{itd} = E_{td}\left[(1 + \tilde{R}^d_{i,t+1})/(1 + \tilde{R}^d_{b,t+1})\right]. \quad (2a)$$

In order to simplify the notation, we can ignore the letter d, and write

$$1 + \mu_{it} = E_t\left[(1 + \tilde{R}_{i,t+1})/(1 + \tilde{R}_{b,t+1})\right]. \quad (2b)$$

[1] In this manuscript, the tilde (~) indicates a random variable. Operators E_t, V_t, and Cov_t refer respectively to mathematical expectations, variance and covariance, where index t implies that we consider the available information at time t (index k refers to investor k).

For the lowest-risk asset (or portfolio) identified by l, we also have

$$1 + \mu_{lt} = E_t\left[(1 + \tilde{R}_{l,t+1})/(1 + \tilde{R}_{b,t+1})\right]. \quad (3)$$

Equation (2) minus (3), indicates that

$$\phi_{it} = E_t\left[(\tilde{R}_{i,t+1} - \tilde{R}_{l,t+1})/(1 + \tilde{R}_{b,t+1})\right], \quad (4a)$$

with $\phi_{it} \equiv \mu_{it} - \mu_{lt}$. Using a compact formulation, we can express

$$\phi_{it} = E_t\left[\tilde{F}_{t+1}\tilde{r}_{i,t+1}\right], \quad (4b)$$

where $\tilde{F}_{t+1} \equiv (1 + \tilde{R}_{b,t+1})^{-1}$ represents the benchmark factor at time $t + 1$, and $\tilde{r}_{i,t+1} \equiv \tilde{R}_{i,t+1} - \tilde{R}_{l,t+1}$ is the excess return of asset i, at time $t + 1$. This benchmark factor ($\tilde{F}_{t+1}$) is similar (in its form) to the familiar *stochastic discount factor* (*SDF*). However, our benchmark factor is not based on a pure *Arrow-Debreu security,* and on *the law of one price*. In fact, equation 1 is not necessarily equal to *one*, and equation 4 is not necessarily equal to *zero*. In this regard, compared to the classic arbitrage approach, as well presented in Campbell (2018, Chapter 4), our model allows us to avoid the following restrictive assumptions: (1) there exists an unrealistic *Arrow-Debreu security*, and (2) arbitrage opportunities are not possible. This represents an important characteristic of our model considering, as noted by Campbell (2018, p. 112), that *arbitrage opportunities are frequently visible in financial markets*. From equation 4b the covariance implies that

$$\phi_{it} = Cov_t\left[\tilde{F}_{t+1}, \tilde{r}_{i,t+1}\right] + E_t[\tilde{F}_{t+1}]E_t\left[\tilde{r}_{i,t+1}\right]. \quad (5)$$

Integrating (5) in (4b) we can write

$$1 = E_t\left[\tilde{F}_{t+1}\tilde{r}_{i,t+1}/\phi_{it}\right] = E_t\left[\tilde{F}_{t+1}\tilde{Y}_{i,t+1}\right], \quad (6)$$

where $\tilde{Y}_{i,t+1} \equiv \frac{\tilde{r}_{i,t+1}}{Cov_t[\tilde{F}_{t+1},\tilde{r}_{i,t+1}]+E_t[\tilde{F}_{t+1}]E_t[\tilde{r}_{i,t+1}]}$. For the benchmark portfolio

$$1 = E_t\left[\tilde{F}_{t+1}\tilde{Y}_{b,t+1}\right]. \quad (7)$$

Thus, equation (6) minus (7) indicates

$$0 = E_t\left[\tilde{F}_{t+1}(\tilde{Y}_{i,t+1} - \tilde{Y}_{b,t+1})\right], \quad (8)$$

and the mathematical definition of covariance shows that

$$Cov_t\left[\tilde{F}_{t+1}, \tilde{Y}_{i,t+1} - \tilde{Y}_{b,t+1}\right] = -E_t[\tilde{F}_{t+1}]E_t\left[\tilde{Y}_{i,t+1} - \tilde{Y}_{b,t+1}\right], \quad (9)$$

or after simple manipulations

$$Cov_t\left[\tilde{F}_{t+1}, \tilde{Y}_{i,t+1} - \tilde{Y}_{b,t+1}\right] = E_t[\tilde{F}_{t+1}]E_t\left[\tilde{Y}_{b,t+1}\right] - E_t[\tilde{F}_{t+1}]E_t\left[\tilde{Y}_{i,t+1}\right]. \quad (10)$$

Isolating the expected value of $\tilde{Y}_{i,t+1}$ indicates

$$E_t\left[\tilde{Y}_{i,t+1}\right] = E_t\left[\tilde{Y}_{b,t+1}\right] - Cov_t\left[\tilde{F}_{t+1}, \tilde{Y}_{i,t+1} - \tilde{Y}_{b,t+1}\right]/E_t[\tilde{F}_{t+1}]. \quad (11)$$

Using the properties of covariance yields

$$E_t\left[\tilde{Y}_{i,t+1}\right] = E_t\left[\tilde{Y}_{b,t+1}\right] + Cov_t\left[\tilde{F}_{t+1}, \tilde{Y}_{b,t+1}\right]/E_t[\tilde{F}_{t+1}] - Cov_t\left[\tilde{F}_{t+1}, \tilde{Y}_{i,t+1}\right]/E_t[\tilde{F}_{t+1}]. \quad (12)$$

Multiplying by the denominator of variable $\tilde{Y}_{i,t+1}$ on each side allows us to write

$$E_t\left[\tilde{r}_{i,t+1}\right] = E_t\left[\tilde{Y}_{b,t+1}\right](Cov_t\left[\tilde{F}_{t+1}, \tilde{r}_{i,t+1}\right] + E_t[\tilde{F}_{t+1}]E_t\left[\tilde{r}_{i,t+1}\right]) + Cov_t\left[\tilde{F}_{t+1}, \tilde{Y}_{b,t+1}\right](Cov_t\left[\tilde{F}_{t+1}, \tilde{r}_{i,t+1}\right] + E_t[\tilde{F}_{t+1}]E_t\left[\tilde{r}_{i,t+1}\right])/E_t[\tilde{F}_{t+1}] - Cov_t\left[\tilde{F}_{t+1}, \tilde{r}_{i,t+1}\right]/E_t[\tilde{F}_{t+1}]. \quad (13)$$

Developing, we can also write

$$E_t\left[\tilde{r}_{i,t+1}\right] = E_t\left[\tilde{Y}_{b,t+1}\right]Cov_t\left[\tilde{F}_{t+1}, \tilde{r}_{i,t+1}\right] + E_t\left[\tilde{Y}_{b,t+1}\right]E_t[\tilde{F}_{t+1}]E_t\left[\tilde{r}_{i,t+1}\right] + Cov_t\left[\tilde{F}_{t+1}, \tilde{Y}_{b,t+1}\right]Cov_t\left[\tilde{F}_{t+1}, \tilde{r}_{i,t+1}\right]/E_t[\tilde{F}_{t+1}] + Cov_t\left[\tilde{F}_{t+1}, \tilde{Y}_{b,t+1}\right]E_t\left[\tilde{r}_{i,t+1}\right] - Cov_t\left[\tilde{F}_{t+1}, \tilde{r}_{i,t+1}\right]/E_t[\tilde{F}_{t+1}]. \quad (14)$$

Regrouping the elements of (14) shows

$$E_t\left[\tilde{r}_{i,t+1}\right] = (E_t\left[\tilde{Y}_{b,t+1}\right]E_t[\tilde{F}_{t+1}] + Cov_t\left[\tilde{F}_{t+1}, \tilde{Y}_{b,t+1}\right])E_t\left[\tilde{r}_{i,t+1}\right] + (E_t\left[\tilde{Y}_{b,t+1}\right] + Cov_t\left[\tilde{F}_{t+1}, \tilde{Y}_{b,t+1}\right]/E_t[\tilde{F}_{t+1}] - 1/E_t[\tilde{F}_{t+1}])Cov_t\left[\tilde{F}_{t+1}, \tilde{r}_{i,t+1}\right], \quad (15)$$

or, if you prefer

$$E_t\left[\tilde{r}_{i,t+1}\right]\left(1 - E_t\left[\tilde{Y}_{b,t+1}\right]E_t[\tilde{F}_{t+1}] - Cov_t\left[\tilde{F}_{t+1}, \tilde{Y}_{b,t+1}\right]\right) = (E_t\left[\tilde{Y}_{b,t+1}\right] + (Cov_t\left[\tilde{F}_{t+1}, \tilde{Y}_{b,t+1}\right] - 1)/E_t[\tilde{F}_{t+1}])Cov_t\left[\tilde{F}_{t+1}, \tilde{r}_{i,t+1}\right]. \quad (16a)$$

To simplify the notation, we may also write

$$E_t\left[\tilde{r}_{i,t+1}\right](H_t) = \\ (E_t\left[\tilde{Y}_{b,t+1}\right] + (Cov_t\left[\tilde{F}_{t+1}, \tilde{Y}_{b,t+1}\right] - 1)/E_t[\tilde{F}_{t+1}])Cov_t\left[\tilde{F}_{t+1}, \tilde{r}_{i,t+1}\right], \quad (16b)$$

with $H_t \equiv 1 - E_t\left[\tilde{Y}_{b,t+1}\right]E_t[\tilde{F}_{t+1}] - Cov_t\left[\tilde{F}_{t+1}, \tilde{Y}_{b,t+1}\right]$. Multiplying by $E_t[\tilde{F}_{t+1}]$ on each side of equation (16b), we have

$$E_t\left[\tilde{r}_{i,t+1}\right](H_t)E_t[\tilde{F}_{t+1}] = \\ (E_t\left[\tilde{Y}_{b,t+1}\right]E_t[\tilde{F}_{t+1}] + Cov_t\left[\tilde{F}_{t+1}, \tilde{Y}_{b,t+1}\right] - 1)Cov_t\left[\tilde{F}_{t+1}, \tilde{r}_{i,t+1}\right], \quad (17a)$$

or equivalently

$$E_t\left[\tilde{r}_{i,t+1}\right](H_t)E_t[\tilde{F}_{t+1}] = (-1)(H_t)Cov_t\left[\tilde{F}_{t+1}, \tilde{r}_{i,t+1}\right]. \quad (17b)$$

From equations (7) it is easy to deduce that H_t equals zero, and that equation (17b) becomes degenerate ($0 = 0$). To overcome this issue, we use a standard regularisation procedure which captures the idea that we work in different economies close to the initial degenerate case, while maintaining a well-defined system. More precisely, following Judd (1998) and Mennuni et al. (2024), we use a perturbation technique based on the *continuity principle*, defined in this way: *small perturbations in the economic environment should not lead to discontinuous changes in the relevant economic relations*.

First, we suppose the existence of a small perturbation parameter, $\varepsilon > 0$, and proceed in a sequence of local perturbed economies where H_t is substituted by ε, while all other variables remain unchanged. This procedure allows us to express

$$E_t\left[\tilde{r}_{i,t+1}\right](\varepsilon)E_t[\tilde{F}_{t+1}] = (-1)(\varepsilon)Cov_t\left[\tilde{F}_{t+1}, \tilde{r}_{i,t+1}\right]. \quad (18a)$$

Second, we divide by ε on both sides of equation (18a), and write

$$E_t\left[\tilde{r}_{i,t+1}\right](\varepsilon/\varepsilon)E_t[\tilde{F}_{t+1}] = (-1)(\varepsilon/\varepsilon)Cov_t\left[\tilde{F}_{t+1}, \tilde{r}_{i,t+1}\right]. \quad (18b)$$

Third, in conformity with the continuity principle, we take the limit as $\varepsilon \to 0$ to return to the initial case (with $\varepsilon = 0$), which suggests the following equality

$$\lim_{\varepsilon \to 0} E_t\left[\tilde{r}_{i,t+1}\right](\varepsilon/\varepsilon)E_t[\tilde{F}_{t+1}] = \lim_{\varepsilon \to 0}(-1)(\varepsilon/\varepsilon)Cov_t\left[\tilde{F}_{t+1}, \tilde{r}_{i,t+1}\right]. \quad (19a)$$

or, after manipulations

$$E_t\left[\tilde{r}_{i,t+1}\right]E_t[\tilde{F}_{t+1}] = -Cov_t\left[\tilde{F}_{t+1}, \tilde{r}_{i,t+1}\right]. \quad (19b)$$

The previous step reflects the idea that economically meaningful relationships are preserved under small perturbations. Solving for the expected excess return, we get

$$E_t\left[\tilde{r}_{i,t+1}\right] = -(1/E_t[\tilde{F}_{t+1}])Cov_t\left[\tilde{F}_{t+1}, \tilde{r}_{i,t+1}\right]. \quad (20)$$

For the benchmark portfolio, we also get

$$E_t\left[\tilde{r}_{b,t+1}\right] = -(1/E_t[\tilde{F}_{t+1}])\left[\tilde{F}_{t+1}, \tilde{r}_{b,t+1}\right]. \quad (21)$$

Introducing (21) in (20) indicates

$$E_t\left[\tilde{r}_{i,t+1}\right] = E_t\left[\tilde{r}_{b,t+1}\right]Cov_t\left[\tilde{F}_{t+1}, \tilde{r}_{i,t+1}\right]/Cov_t\left[\tilde{F}_{t+1}, \tilde{r}_{b,t+1}\right]. \quad (22)$$

Using our definition of the excess return yields

$$E_t\left[\tilde{R}_{i,t+1}\right] = E_t\left[\tilde{R}_{l,t+1}\right] + \left(E_t\left[\tilde{R}_{b,t+1}\right] - E_t\left[\tilde{R}_{l,t+1}\right]\right)\frac{Cov_t\left[\tilde{F}_{t+1}, \tilde{r}_{i,t+1}\right]}{Cov_t\left[\tilde{F}_{t+1}, \tilde{r}_{b,t+1}\right]}, \quad (23)$$

or to simplify the notation (as in Campbell, 2018, page 94)

$$E_t\left[\tilde{R}_{i,t+1}\right] = E_t\left[\tilde{R}_{l,t+1}\right] + \lambda_t \mathrm{B}_{Bit}, \quad (24)$$

$$\lambda_t \equiv E_t\left[\tilde{R}_{b,t+1}\right] - E_t\left[\tilde{R}_{l,t+1}\right],$$

$$\mathrm{B}_{Bit} \equiv Cov_t\left[\tilde{F}_{t+1}, \tilde{R}_{i,t+1} - \tilde{R}_{l,t+1}\right]/Cov_t\left[\tilde{F}_{t+1}, \tilde{R}_{b,t+1} - \tilde{R}_{l,t+1}\right].$$

Equation (24) represents our first and main result. This equation reveals that the expected return of an asset is equal to the expected return of the lowest-risk asset (or portfolio), plus a risk premium directly proportional to a benchmark beta, obtained from the covariance between the asset's excess return and the benchmark factor.

In (24), assumption A2 (*risk aversion*) implies that the risk premium corresponds to $\lambda_t \mathrm{B}_{Bit}$. Here, we interpret the *parameter lambda* (λ_t) as the *price of risk*, and the *benchmark beta* (B_{Bit}) as the *quantity of risk*. For the lowest-risk asset (noted by l), the excess returns equal zero, just like the quantity of risk, and (24) predicts that: $E_t\left[\tilde{R}_{i,t+1}\right] = E_t\left[\tilde{R}_{l,t+1}\right]$. Also, for the benchmark portfolio (noted by b), the quantity of risk corresponds to *one*, and (24) now predicts that: $E_t\left[\tilde{R}_{i,t+1}\right] = E_t\left[\tilde{R}_{b,t+1}\right]$. If the quantity of risk is

superior (inferior) to one, then the required return of the asset is superior (inferior) to the expected return of the benchmark portfolio.

In short, we have shown that the relationship between the risk of an asset and its expected return can be characterized using only three basic conditions, expressed by the unrestrictive assumptions above. This relationship is linear, just like the CAPM (and its most important extensions). Besides, we can take a Taylor approximation of (24) to express betas in terms of a more concrete variable and to express the linear relationship with a useful prediction closer to the classic CAPM.

A special case: The classic CAPM

According to Taylor's theorem, we can use an approximation around the point $\bar{R}_{bt}$ to obtain

$$f\big(\tilde{R}_{b,t+1}\big) \approx f(\bar{R}_{bt}) + f'(\bar{R}_{bt})\big(\tilde{R}_{b,t+1} - \bar{R}_{bt}\big), \tag{25}$$

where $\bar{R}_{bt} \equiv E_t\big[\tilde{R}_{b,t+1}\big]$. Integrating (25) into (23) suggests that the expected excess return of an asset approaches the following value:

$$E_t\big[\tilde{R}_{i,t+1}\big] \approx E_t\big[\tilde{R}_{l,t+1}\big] + \lambda_t \frac{Cov_t\big[f(\bar{R}_{bt}) + f'(\bar{R}_{bt})\big(\tilde{R}_{b,t+1} - \bar{R}_{bt}\big), \tilde{r}_{i,t+1}\big]}{Cov_t\big[f(\bar{R}_{bt}) + f'(\bar{R}_{bt})\big(\tilde{R}_{b,t+1} - \bar{R}_{bt}\big), \tilde{r}_{b,t+1}\big]}. \tag{26}$$

Using the basic properties of covariance gives

$$E_t\big[\tilde{R}_{i,t+1}\big] \approx E_t\big[\tilde{R}_{l,t+1}\big] + \lambda_t \frac{Cov_t\big[\tilde{R}_{b,t+1}, \tilde{r}_{i,t+1}\big]f'(\bar{R}_{bt})}{Cov_t\big[\tilde{R}_{b,t+1}, \tilde{r}_{b,t+1}\big]f'(\bar{R}_{bt})}. \tag{27}$$

Introducing our definition of the excess return shows

$$E_t\big[\tilde{R}_{i,t+1}\big] \approx E_t\big[\tilde{R}_{l,t+1}\big] + \lambda_t \frac{Cov_t\big[\tilde{R}_{b,t+1}, \tilde{R}_{i,t+1} - \tilde{R}_{l,t+1}\big]}{Cov_t\big[\tilde{R}_{b,t+1}, \tilde{R}_{b,t+1} - \tilde{R}_{l,t+1}\big]}, \tag{28}$$

or the following relationship

$$E_t\big[\tilde{R}_{i,t+1}\big] \approx E_t\big[\tilde{R}_{l,t+1}\big] + \lambda_t \beta_{Bit}, \tag{29}$$

$$\beta_{Bit} \equiv Cov_t\big[\tilde{R}_{b,t+1}, \tilde{R}_{i,t+1} - \tilde{R}_{l,t+1}\big] / Cov_t\big[\tilde{R}_{b,t+1}, \tilde{R}_{b,t+1} - \tilde{R}_{l,t+1}\big].$$

Equation (29) represents our second result. This equation now reveals that the expected return of an asset is *approximately* equal to the expected return of the lowest-risk asset, plus a risk premium directly proportional to a benchmark beta (in lower case), obtained from the covariance between the asset's excess return and the benchmark *return*.

If we ignore the approximation and suppose the existence of a free-risk asset (denoted by the letter f), then we have

$$E_t\left[\tilde{R}_{i,t+1}\right] = R_{f,t+1} + \left(E_t\left[\tilde{R}_{b,t+1}\right] - R_{f,t+1}\right)\beta_{Bit}, \tag{30}$$
$$\beta_{Bit} = Cov_t\left[\tilde{R}_{b,t+1}, \tilde{R}_{i,t+1}\right]/V_t\left[\tilde{R}_{b,t+1}\right].$$

Therefore, assuming that the market portfolio represents a potential reference index, we can easily see that the classic CAPM corresponds to a special case of our benchmark model.

The benchmark and the lowest-risk asset

Identification of the proper benchmark asset (for performance management) represents a crucial aspect of our model. In our setup, a benchmark is simply a standard against which the performance of a security, mutual fund, or investment manager can be measured. Benchmark types can be organized into categories by asset class (stocks, bonds, and cash) and then further categorized by capitalization ranges, sectors, countries or others. For example, in the U.S. equity market, we can take the usual *S&P 500* (for a large-cap benchmark). In addition, we can take the *S&P 600 SmallCap Index*, *S&P 400 MidCap Index*, *S&P 1500 Composite Index*, or *Russell 3000 Index*. On the international markets, the *MSCI World Index* or *MSCI Emerging Markets Index* can be used. In fixed income, examples of benchmarks include the *Barclays Capital U.S. Treasury Bond Index*, *Barclays Capital U.S. Aggregate Bond Index*, and *FTSE World Government Bond Index* (WGBI).

In the same manner, there is a certain flexibility in the identification of the lowest-risk asset. This asset could be a standard Treasury bill, but in our framework, it does not have to be totally without risk. In fact, in our set-up, there are *N* different risky assets, and we can calculate the expected return for each one. The *lowest-risk asset* is simply the one with the lowest expected return (or lowest rate), among all the *N* assets. Moreover, the covariance between the asset's return and the benchmark is not necessarily zero. Besides, we can refer to an ordinary savings account to estimate the lowest-risk rate, as in Platen (2006) and Platen et al. (2006).

3. Conclusion

In this note, we used the relative return to a benchmark and three unrestrictive conditions to demonstrate that an asset's expected return is equal to the expected return of the lowest-risk asset, plus a risk premium directly proportional to the covariance between the asset's excess return and the benchmark factor. Furthermore, we showed that this linear relationship can be approximated, using Taylor's theorem. Moreover, we proposed that the classic CAPM can be viewed as a special case of our benchmark extension model. Finally, we suggested different ways to identify the benchmark index and lowest-risk rate.

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Appendix A

In this appendix, we demonstrate the theoretical robustness of the model by showing that its main result can be obtained without any reference to regularisation arguments. To do

this, we first introduce the following assumption: *the relative returns on any asset tend to be equal to the benchmark rate in the economy, plus a disturbance term with zero mean.*

For clarity, let $\tilde{R}_{i,t+1}^{Relative}$ be the relative returns of asset i at time $t+1$, with $\tilde{R}_{i,t+1}^{Relative} \equiv (1+\tilde{R}_{i,t+1})/(1+\tilde{R}_{b,t+1}) - 1$. For the benchmark portfolio, represented by the letter b, we thus have: $\tilde{R}_{b,t+1}^{Relative} \equiv (1+\tilde{R}_{b,t+1})/(1+\tilde{R}_{b,t+1}) - 1 = 0$.

Adopting the previous notation and given the available information at time t, we assume the validity of this *benchmark process*

$$\tilde{R}_{i,t+1}^{Relative} = \tilde{R}_{b,t+1}^{Relative} + \tilde{e}_{i,t+1} = \tilde{e}_{i,t+1}\,, \tag{A1}$$

$$E_t[\tilde{e}_{i,t+1}] = 0, \tag{A2}$$

where $\tilde{e}_{i,t+1}$ is the usual random error term. Taking the expected value of each side of equation (A1), we have

$$E_t\left[\tilde{R}_{i,t+1}^{Relative}\right] = 0. \tag{A3}$$

Our *benchmark process* simply suggests that, for different states of nature and probabilities, the relative returns of an asset (such as a stock) will sometimes be higher and sometimes lower to the corresponding benchmark rate in the economy. Since the relative return on the benchmark portfolio is equal to zero, the corresponding relative return on any stock should fluctuate around 0%. This point of view is fully consistent with the risk-aversion postulate and a positive relationship between risk and return, described in nominal terms. Now, using the notation above in our initial equation (2b), we can write

$$1 + E_t\left[\tilde{R}_{i,t+1}^{Relative}\right] = E_t\left[(1+\tilde{R}_{i,t+1})/(1+\tilde{R}_{b,t+1})\right], \tag{A4}$$

and accepting our benchmark process, equation (A4) becomes

$$1 = E_t\left[(1+\tilde{R}_{i,t+1})/(1+\tilde{R}_{b,t+1})\right], \tag{A5}$$

or, put it differently

$$1 = E_t\left[\tilde{F}_{t+1}(1+\tilde{R}_{i,t+1})\right]. \tag{A6}$$

For the lowest-risk asset, included in the benchmark portfolio, we also have

$$1 = E_t\left[\tilde{F}_{t+1}(1 + \tilde{R}_{l,t+1})\right], \tag{A7}$$

while equation (A6) minus (A7) reveals that

$$0 = E_t\left[\tilde{F}_{t+1}(\tilde{R}_{i,t+1} - \tilde{R}_{l,t+1})\right], \tag{A8}$$

or, using our notation

$$0 = E_t\left[\tilde{F}_{t+1}\tilde{r}_{i,t+1}\right]. \tag{A9}$$

Equation (A9) and the covariance definition yields

$$Cov_t\left[\tilde{F}_{t+1}, \tilde{r}_{i,t+1}\right] = -E_t[\tilde{F}_{t+1}]E_t\left[\tilde{r}_{i,t+1}\right]. \tag{A10}$$

Therefore, equation (A10) allows us to recover equation (20) and our main result, expressed by equation (24), without any reference to regularisation arguments.