Research Article

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Eigenvalue regions and the minimal dimension of realization for structured classes of stochastic matrices

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Abstract: Given a class $\mathcal{X}$ of stochastic matrices and a complex number λ, we ask for the smallest dimension of a matrix in the class having λ as an eigenvalue. We study this problem for several structured classes. For circulant stochastic matrices, the eigenvalue regions are regular polygons, from which we obtain a geometric formula for the minimal realization dimension. We also solve the problem for anticirculant matrices, convex combinations of the identity and a permutation matrix, and convex combinations of powers of a permutation. The latter class connects the realization problem with the Perfect–Mirsky conjecture. For tridiagonal Toeplitz-type stochastic matrices, we show that the eigenvalue region in dimension n is $[-\cos(\pi/n), 1]$ and derive an explicit realization formula. Finally, for pentadiagonal Toeplitz-type matrices, we obtain exact results in dimensions three and four and exhibit a nonreal eigenvalue whose minimal realization dimension is five.

Keywords: stochastic matrix; eigenvalue region; minimal dimension of realization; Perfect–Mirsky region; Karpelevič region

MSC 2020: 15B51; 15A18; 15B05; 60J10

1 Introduction

1.1 The eigenvalue regions of stochastic matrices

A (row) *stochastic matrix* is a square matrix $S = (s_{ij})$ with nonnegative entries whose rows sum to one:

$$s_{ij} \geq 0, \qquad \sum_{j=1}^{n} s_{ij} = 1, \qquad (1 \leq i, j \leq n).$$

Such matrices arise naturally in probability theory, where they represent the transition operators of finite Markov chains, and their spectral properties govern the long-term behavior, mixing rates, and periodicity of

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these chains. Throughout the paper, $\sigma(S)$ denotes the *spectrum* of a square matrix S, that is, the set of its eigenvalues, and

$$\overline{\mathbb{D}} := \{z \in \mathbb{C} : \ |z| \leq 1\}$$

denotes the closed unit disk.

Since every eigenvalue of a stochastic matrix lies in $\overline{\mathbb{D}}$, a natural problem is to determine which complex numbers in the disk can actually occur as eigenvalues. This question originated in a seminar of Kolmogorov on Markov chains at Moscow State University and was formulated in 1938 as the problem of describing the possible eigenvalues of stochastic matrices [1,2]. It has since become one of the classical problems in the spectral theory of nonnegative matrices. For each $n \geq 1$, let

$$\Theta_n := \{\lambda \in \mathbb{C} : \ \lambda \in \sigma(S) \text{ for some } n \times n \text{ stochastic matrix } S\}.$$

Substantial early progress was obtained in the mid-1940s by Dmitriev and Dynkin [3,4], who developed a geometric approach to the location of eigenvalues and obtained a partial description of the regions Θ_n. A few years later, Karpelevič completed the program by showing that the boundary of Θ_n consists of a finite collection of algebraic arcs joining appropriate roots of unity [5]; the resulting spectral regions are now known as the *Karpelevič regions*. For every $n \geq 2$, the region Θ_n is star-shaped with respect to the origin; the regions form a nested sequence

$$\Theta_1 \subseteq \Theta_2 \subseteq \cdots$$

(if $\lambda \in \Theta_n$, then adjoining a one-dimensional stochastic block shows that $\lambda \in \Theta_{n+1}$), and

$$\overline{\bigcup_{n \geq 1} \Theta_n} = \overline{\mathbb{D}}.$$

Although the existence of the boundary arcs has been known for many decades, their detailed structure remains surprisingly subtle. A considerable amount of recent work has focused on revisiting Karpelevič's original proof, clarifying the geometry of the boundary, constructing realizing matrices for individual arcs, and obtaining more explicit parametrizations of the extremal points of Θ_n; see [6–12] and the references therein.

1.2 Structured subclasses

A closely related and intensively studied line of research concerns the analogous questions for distinguished subclasses of stochastic matrices. The most prominent example is the *doubly stochastic single eigenvalue problem*, which asks for the region DS_n of eigenvalues of $n \times n$ doubly stochastic matrices. Perfect and Mirsky [13] observed that DS_n contains the union

$$\mathrm{PM}_n := \bigcup_{k \leq n} \Pi_k$$

of the convex hulls Π_k of the k-th roots of unity, verified that equality holds for $n \leq 3$, and conjectured that $\mathrm{DS}_n = \mathrm{PM}_n$ for every n. The conjecture was confirmed for $n = 4$ by Levick, Pereira and Kribs [14], while Mashreghi and Rivard [15] showed that it fails for $n = 5$. Extensive computational investigations of the regions DS_n, together with the conjecture that every boundary point of DS_n is attained by a convex combination of at most two permutation matrices, appear in [16]; see also [9]. Eigenvalue localization results are also available for further subclasses, such as trace-zero doubly stochastic matrices [17].

Several further subclasses bear directly on the classes studied below, and it is worth indicating in each case what is already known and where the present paper departs from it. Jankowski, Johnson and Lim [18] determined the *hull spectra* of abelian matrix groups, that is, the sets of eigenvalues of convex combinations of the elements of such a group; convex combinations of the powers of a single permutation matrix fall within their framework, and the identification of the corresponding regions with the Perfect–Mirsky regions, which we recover in Section 4 by a short direct argument, is a special case of their results. Banded matrices have also been considered, but a bandwidth restriction alone has no effect whatsoever on the real eigenvalue regions: every

point of $[-1,1]$ is already an eigenvalue of a tridiagonal doubly stochastic matrix in every dimension $n \geq 2$, and of a symmetric pentadiagonal doubly stochastic matrix in every dimension $n \geq 3$ (see [19, Propositions 1 and 5]; a 2×2 symmetric block padded by an identity block suffices). The dimension-dependent regions obtained in Sections 5 and 6 therefore rest on a further homogeneity assumption, under which the matrices become Toeplitz up to a modification of their boundary rows. For such matrices of bandwidth one, the eigenvalues are known explicitly, thanks to a line of work on tridiagonal Toeplitz matrices with perturbed corner entries [20–23], and our contribution in that case lies in the description of the regions and of the realization function rather than in the spectral formula itself. With bandwidth two the situation is different: the characteristic polynomials of symmetric pentadiagonal Toeplitz matrices can be expressed through Chebyshev polynomials [24], their eigenvalue asymptotics are understood [25], and determinantal recursions are available for general pentadiagonal matrices [26], but the stochastic boundary convention adopted below produces nonsymmetric matrices, with possibly nonreal eigenvalues, to which none of these results applies directly, and the low-dimensional analysis of Section 6 proceeds from first principles.

Finally, eigenvalue regions have recently been determined under other kinds of structural restriction: nonnegative matrices with prescribed zero patterns [27], monotone stochastic matrices [28], and stochastic matrices supported on a cycle [29,30]. In each of these cases the restriction is strong enough for the region to be described explicitly in low dimension, which is also the situation exploited in the present paper, and the minimal dimension of realization question raised here could be asked of these classes as well.

1.3 The minimal dimension of realization

The present paper considers a complementary question. Since the Karpelevič regions are nested, one may ask not only whether a complex number λ belongs to some region Θ_n, but also for the *smallest* dimension in which it can occur. This point of view is already implicit in the work of Dmitriev and Dynkin, who considered the new spectral points appearing in the difference $\Theta_{n+1} \setminus \Theta_n$. It leads to a realization problem that appears to have received relatively little systematic attention: given $\lambda \in \overline{\mathbb{D}}$, determine the smallest positive integer n such that $\lambda \in \Theta_n$.

If $\lambda = re^{i\theta}$ is an eigenvalue of a transition matrix, the corresponding spectral mode decays at rate r and changes phase by θ at each step. The smallest dimension in which λ can be realized within a class $\mathcal{X}$ is therefore the size of the smallest state space on which a chain of the prescribed type can exhibit this spectral behavior. Similar minimality questions arise in neighboring inverse problems. In the nonnegative inverse eigenvalue problem, for example, the theorem of Boyle and Handelman [31] characterizes the nonzero spectra of nonnegative matrices after sufficiently many zero eigenvalues are added. Determining the smallest number of zeros that must be added is itself a subtle problem.

1.4 Classes studied and main results

We study the realization problem for several structured classes, which we present in order of increasing difficulty. After setting up the problem in general in Section 2, we begin in Section 3 with the circulant stochastic matrices, whose eigenvalue regions are the polygons Π_n spanned by the n-th roots of unity; the geometry of these polygons yields a radial formula for the minimal realization dimension, together with an explicit upper bound for it. The anticirculant stochastic matrices, by contrast, have the region $[-1,1]$ in every dimension $n \geq 2$. We also show that the stochastic Toeplitz and Hankel matrices coincide with the circulant and anticirculant classes, respectively, so that four classical structure classes are disposed of at once.

Section 4 is devoted to two classes generated by a single permutation matrix: the convex combinations of the identity and a permutation, and the convex combinations of the powers of a permutation. These classes connect the circulant case with the doubly stochastic single eigenvalue problem. The first leads to a star of chords and to a realization function of arithmetic nature, while the second realizes the nested Perfect–Mirsky regions without changing the circulant realization function; as a by-product, the Perfect–Mirsky conjecture itself can be restated as the equality of two realization functions.

Sections 5 and 6 concern classes defined by a locality condition. For the tridiagonal Toeplitz-type stochastic matrices, which represent homogeneous birth–death chains on a finite path, we determine the eigenvalue regions and the realization function in closed form. Allowing jumps of length two leads to pentadiagonal Toeplitz-type matrices, which may have nonreal eigenvalues; for these we obtain exact results in dimensions three and four and exhibit an explicit nonreal eigenvalue whose minimal realization dimension is five. Section 7 returns to the realization problem for the full class of stochastic matrices and reformulates it in terms of the radial boundary functions of the Karpelevič regions.

2 The minimal dimension of realization problem

Let $\mathcal{X}$ be a class of stochastic matrices, by which we mean a collection of stochastic matrices of all orders. Given a positive integer $n \geq 1$, define

$$\Theta_n(\mathcal{X}) := \{\lambda \in \mathbb{C} : \ \lambda \in \sigma(S) \text{ for some } n \times n \text{ matrix } S \in \mathcal{X}\},$$

and set

$$\Theta(\mathcal{X}) := \bigcup_{n \geq 1} \Theta_n(\mathcal{X}).$$

The regions $\Theta_n(\mathcal{X})$ need not form a nested sequence. They do whenever the class $\mathcal{X}$ is closed under the operation

$$S \longmapsto \begin{pmatrix} S & 0 \\ 0 & 1 \end{pmatrix},$$

which amounts to adjoining an additional absorbing state to the corresponding Markov chain; in that case one obtains a nested hierarchy of spectral regions analogous to the classical hierarchy of Karpelevič regions. Nothing in the following definition, however, assumes nestedness.

Definition 2.1. Let $\mathcal{X}$ be a class of stochastic matrices. The *minimal dimension of realization function* associated to $\mathcal{X}$,

$$d_{\mathcal{X}} : \mathbb{C} \to \mathbb{N} \cup \{\infty\},$$

is defined by

$$d_{\mathcal{X}}(\lambda) = \min\{n \geq 1 : \ \lambda \in \Theta_n(\mathcal{X})\},$$

with the convention that

$$d_{\mathcal{X}}(\lambda) = \infty$$

whenever $\lambda \notin \Theta(\mathcal{X})$.

Thus $d_{\mathcal{X}}(\lambda)$ is the smallest dimension in which the eigenvalue λ can be realized by a matrix belonging to the class $\mathcal{X}$. Since every eigenvalue of a stochastic matrix lies in $\overline{\mathbb{D}}$, one has $\Theta(\mathcal{X}) \subseteq \overline{\mathbb{D}}$ for every class $\mathcal{X}$, so that $d_{\mathcal{X}}$ is identically ∞ on $\mathbb{C} \setminus \overline{\mathbb{D}}$; the explicit formulas obtained in this paper describe the realization functions on $\overline{\mathbb{D}}$, and are tacitly completed off the closed disk by this observation. We also caution that the phrase “minimal realization” has an established and unrelated meaning in systems theory, where it refers to the smallest state-space dimension of a realization of a prescribed transfer function or stochastic process; the notion studied here concerns a single prescribed eigenvalue.

Remark 2.2. The function $d_{\mathcal{X}}$ is well defined for an arbitrary class $\mathcal{X}$, but its meaning depends on whether the regions are nested. When $\Theta_n(\mathcal{X}) \subseteq \Theta_{n+1}(\mathcal{X})$ for all n, the quantity $d_{\mathcal{X}}(\lambda)$ is a genuine threshold: $\lambda \in \Theta_n(\mathcal{X})$ for *every* $n \geq d_{\mathcal{X}}(\lambda)$. Without nestedness, membership may fail in dimensions larger than $d_{\mathcal{X}}(\lambda)$, so that $d_{\mathcal{X}}(\lambda)$ records the first occurrence of λ in the hierarchy rather than a threshold. Both situations arise in this paper.

The *minimal dimension of realization problem* for a class $\mathcal{X}$, which is the principal problem considered in this paper, is therefore to determine the regions $\Theta_n(\mathcal{X})$, their union $\Theta(\mathcal{X})$, and the function $d_{\mathcal{X}}$. We conclude this section with a few examples illustrating the different forms the problem may take.

For the class Σ of all stochastic matrices, the regions $\Theta_n(\Sigma) = \Theta_n$ are the Karpelevič regions described in the introduction. The corresponding minimal dimension of realization function does not seem to have been investigated systematically; we return to this case in Section 7.

Similarly, one may consider the classes of doubly stochastic, unistochastic, and orthostochastic matrices. For the doubly stochastic class, determining the realization function is equivalent to determining all of the regions DS_n of the single eigenvalue problem discussed in the introduction, which is open for $n \geq 5$ [13–16]; for the unistochastic and orthostochastic classes, even less is known. Remarks 3.10 and 4.9 below record what the classes studied in this paper say about the doubly stochastic realization function.

Another natural example is provided by the convex combinations of two permutation matrices, whose significance stems from the boundary conjecture of [16] recalled in the introduction. Computational evidence suggests that the associated spectral regions are assembled from families of algebraic arcs connecting roots of unity; despite this comparatively rigid geometry, neither the regions nor the corresponding minimal dimension of realization function seem to be understood in general. In Section 4 we solve the realization problem completely for the subclass in which one of the two permutations is the identity, as well as for the related class of convex combinations of the powers of a single permutation.

For comparison, the realization problem is elementary in the following three classes.

Consider first the class $\mathcal{U}$ of all upper triangular stochastic matrices. The eigenvalues of a matrix in $\mathcal{U}$ are its diagonal entries, which belong to $[0, 1]$; conversely, every $\lambda \in [0, 1]$ is an eigenvalue of the upper triangular stochastic matrix

$$\begin{pmatrix} \lambda & 1-\lambda \\ 0 & 1 \end{pmatrix}.$$

Hence $\Theta_n(\mathcal{U}) = [0, 1]$ for every $n \geq 2$: the spectral hierarchy stabilizes at dimension 2, and

$$d_{\mathcal{U}}(1) = 1, \qquad d_{\mathcal{U}}(\lambda) = 2 \quad (0 \leq \lambda < 1), \qquad d_{\mathcal{U}}(\lambda) = \infty \quad (\lambda \in \overline{\mathbb{D}} \setminus [0, 1]).$$

Let now $\mathcal{S}$ denote the class of symmetric stochastic matrices. Every matrix in $\mathcal{S}$ has real spectrum, contained in $[-1, 1]$; conversely, for every $\lambda \in [-1, 1]$, the symmetric stochastic matrix

$$\begin{pmatrix} \frac{1+\lambda}{2} & \frac{1-\lambda}{2} \\ \frac{1-\lambda}{2} & \frac{1+\lambda}{2} \end{pmatrix}$$

has eigenvalues 1 and λ. Hence the spectral hierarchy again stabilizes at dimension 2, and

$$d_{\mathcal{S}}(1) = 1, \qquad d_{\mathcal{S}}(\lambda) = 2 \quad (-1 \leq \lambda < 1), \qquad d_{\mathcal{S}}(\lambda) = \infty \quad (\lambda \in \overline{\mathbb{D}} \setminus [-1, 1]).$$

Finally, let $\mathcal{P}$ denote the class of all permutation matrices. Every permutation matrix decomposes into disjoint cycles, and a cycle of length ℓ contributes to the spectrum all the ℓ-th roots of unity; in particular, the multiplicative order $\mathrm{ord}(\zeta)$ of an eigenvalue ζ (that is, the smallest positive integer m with $\zeta^m = 1$) divides the length of some cycle. Consequently, an eigenvalue ζ occurs for some $n \times n$ permutation matrix if and only if ζ is a root of unity with $\mathrm{ord}(\zeta) \leq n$: necessity holds because $\mathrm{ord}(\zeta)$ divides a cycle length, which is at most n; sufficiency because a cycle of length $\mathrm{ord}(\zeta)$, completed by fixed points, admits every $\mathrm{ord}(\zeta)$-th root of unity (in particular ζ) as an eigenvalue. Thus

$$d_{\mathcal{P}}(\zeta) = \mathrm{ord}\,(\zeta)$$

for every root of unity ζ, while

$$d_{\mathcal{P}}(\lambda) = \infty$$

for every $\lambda \in \overline{\mathbb{D}}$ that is not a root of unity.

The remainder of the paper is devoted to classes lying between these extremes: rich enough for the spectral regions to vary nontrivially with the dimension, yet structured enough for the regions and the realization function to admit explicit geometric descriptions.

3 Circulant and anticirculant stochastic matrices

This section treats circulant and anticirculant stochastic matrices. Although the two families share much of their algebraic structure, their realization theories differ markedly: circulant stochastic matrices give rise to regular polygonal regions, whereas the anticirculant regions collapse to a real interval. Moreover, the two classes absorb two further classical structure classes: we show below that the Toeplitz stochastic matrices are exactly the circulant stochastic matrices (Proposition 3.3) and that the Hankel stochastic matrices are exactly the anticirculant stochastic matrices (Proposition 3.14). The analysis of this section therefore disposes of four structure classes at once.

Throughout this section, the coefficients defining circulant, anticirculant and Hankel matrices are indexed starting from 0, and, where indicated, the row and column indices are taken in $\{0, 1, \dots, n-1\}$ as well.

3.1 Circulant stochastic matrices

We begin with the class $\mathcal{C}$ of circulant stochastic matrices. Recall that an $n \times n$ matrix C is *circulant* if there exist numbers $a_0, a_1, \dots, a_{n-1}$ such that

$$C = \operatorname{Circ}(a_0, a_1, \dots, a_{n-1}) := \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \cdots & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & \cdots & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_0 \end{pmatrix};$$

equivalently, if its (i, j) entry depends only on $j - i$ modulo n. If, in addition, C is stochastic, then

$$a_j \geq 0, \qquad (0 \leq j \leq n-1),$$

and

$$a_0 + a_1 + \cdots + a_{n-1} = 1.$$

The spectral theory of circulant matrices is classical; see, for example, [32]. The identification of the spectral regions in Theorem 3.1 below is essentially implicit in this classical theory; the ingredients may be extracted from [33], and a geometric account was presented in expository form, in French, in [34]. We therefore only recall the main argument, which will be used again in the proof of Theorem 4.6. By contrast, the radial description of the realization function developed in Theorem 3.5, Corollary 3.6 and Proposition 3.7 below, elementary though it is, does not seem to have been recorded before.

Writing

$$\omega_n := \mathrm{e}^{2\pi \mathrm{i}/n},$$

let

$$\Pi_n := \operatorname{conv}\left\{1, \omega_n, \omega_n^2, \dots, \omega_n^{n-1}\right\}$$

denote the regular polygon whose vertices are the n-th roots of unity, as in the introduction (thus $\Pi_1 = \{1\}$ and $\Pi_2 = [-1, 1]$).

Theorem 3.1 (see [32–34]). *For every $n \geq 1$,*

$$\Theta_n(\mathcal{C}) = \Pi_n.$$

Proof. The eigenvalues of $C = \mathrm{Circ}(a_0, \dots, a_{n-1})$ are the values $p\left(\omega_n^k\right)$, $k = 0, \dots, n-1$, at the n-th roots of unity of the polynomial $p(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1}$ [32]. If C is stochastic, then each $p\left(\omega_n^k\right)$ is a convex combination of n-th roots of unity, hence a point of Π_n. Conversely, every $\lambda \in \Pi_n$ can be written as $\lambda = p(\omega_n)$ with nonnegative coefficients $a_0, \dots, a_{n-1}$ summing to 1, and is then an eigenvalue of the stochastic matrix $\mathrm{Circ}(a_0, \dots, a_{n-1})$. □

Remark 3.2. Unlike the Karpelevič regions, the polygons Π_n do not form a nested sequence. For example, the vertex ω_3 of Π_3 lies on the unit circle without being a vertex of Π_4, so that

$$\Pi_3 \not\subseteq \Pi_4,$$

and similarly $\Pi_4 \not\subseteq \Pi_5$. Consequently, the realization problem for circulant stochastic matrices cannot be interpreted in terms of the first stage at which a point enters an increasing hierarchy of spectral regions. Instead, one must determine the smallest regular polygon Π_n containing the point; see Example 3.8.

Before solving this geometric problem, we pause to record that it also governs a second classical structure class. At first sight, Toeplitz stochastic matrices (stochastic matrices with constant diagonals) appear to form a strictly larger class than the circulant stochastic matrices. The stochasticity condition, however, forces the boundary diagonals to satisfy precisely the compatibility relations required for circulant structure.

Proposition 3.3. *For every $n \geq 1$, the class of $n \times n$ Toeplitz stochastic matrices coincides with the class of $n \times n$ circulant stochastic matrices.*

Proof. Every circulant matrix is Toeplitz. Indeed, if $C = \mathrm{Circ}(a_0, \dots, a_{n-1})$, then $C_{ij} = a_{(j-i) \bmod n}$ for $1 \leq i, j \leq n$; this entry depends only on $j - i$, so $C_{ij} = t_{j-i}$ with $t_k := a_{k \bmod n}$ for $-(n-1) \leq k \leq n-1$.

Conversely, let T be an $n \times n$ Toeplitz stochastic matrix, say $T_{ij} = t_{j-i}$ for $1 \leq i, j \leq n$, where $t_{-(n-1)}, \dots, t_{n-1}$ are the defining diagonal values. For $1 \leq i \leq n$, the sum of the entries in the i-th row is

$$\sum_{j=1}^{n} t_{j-i}.$$

Since T is stochastic, these sums are all equal to 1. Hence, for $1 \leq i \leq n-1$,

$$\sum_{j=1}^{n} t_{j-i} = \sum_{j=1}^{n} t_{j-(i+1)}.$$

Expanding the two sums gives

$$t_{1-i} + t_{2-i} + \cdots + t_{n-i} = t_{-i} + t_{1-i} + \cdots + t_{n-i-1}.$$

All common terms cancel, leaving

$$t_{-i} = t_{n-i}, \qquad (1 \leq i \leq n-1).$$

Now set $a_j := t_j$ for $0 \leq j \leq n-1$. If $j \geq i$, then $T_{ij} = t_{j-i} = a_{j-i}$; if $j < i$, then the displayed relations give $T_{ij} = t_{j-i} = t_{j-i+n} = a_{j-i+n}$. In either case $T_{ij} = a_{(j-i) \bmod n}$, that is, $T = \mathrm{Circ}(a_0, \dots, a_{n-1})$. □

Corollary 3.4. *Let $\mathcal{T}$ denote the class of Toeplitz stochastic matrices. For every $n \geq 1$,*

$$\Theta_n(\mathcal{T}) = \Theta_n(C) = \Pi_n, \qquad \text{and hence} \qquad d_{\mathcal{T}} = d_C.$$

Proof. By Proposition 3.3, the two classes coincide, and the claim follows from Theorem 3.1. □

Since $\Theta_n(C) = \Pi_n$, the minimal dimension of realization of a prescribed eigenvalue is determined entirely by the geometry of the polygons Π_n, and it is convenient to work in polar coordinates. Let

$$\lambda = re^{i\theta}, \qquad 0 \le r \le 1.$$

For a fixed value of n, we wish to determine the boundary radius of Π_n in the direction θ.

For $n \ge 3$ and $\theta \in \mathbb{R}$, define

$$\varepsilon_n(\theta) := \min_{k\in\mathbb{Z}} \min_{m\in\mathbb{Z}} \left| \theta - \frac{(2k+1)\pi}{n} + 2\pi m \right|.$$

Thus

$$0 \le \varepsilon_n(\theta) \le \frac{\pi}{n},$$

and $\varepsilon_n(\theta)$ measures the angular distance from the ray of angle θ to the nearest outward normal of a side of Π_n (the outward normals point in the directions $(2k+1)\pi/n$, halfway between consecutive vertices). For $n \ge 3$, let

$$\rho_n(\theta) := \frac{\cos(\pi/n)}{\cos(\varepsilon_n(\theta))}.$$

Theorem 3.5. *Let $n \ge 3$ and let $\lambda = re^{i\theta}$ with $r \ge 0$. Then*

$$\lambda \in \Pi_n \qquad \Longleftrightarrow \qquad r \le \rho_n(\theta).$$

Proof. The sides of Π_n are tangent to the circle of radius $\cos(\pi/n)$ centered at the origin. More precisely, for each $k = 0, \dots, n-1$, the side joining the vertices ω_n^k and ω_n^{k+1} lies on the line

$$x\cos\left(\frac{(2k+1)\pi}{n}\right) + y\sin\left(\frac{(2k+1)\pi}{n}\right) = \cos\left(\frac{\pi}{n}\right).$$

Fix an angle θ and let ρ denote the distance from the origin to the boundary of Π_n along the ray of angle θ. Choose k so that

$$\varepsilon_n(\theta) = \min_{m\in\mathbb{Z}} \left| \theta - \frac{(2k+1)\pi}{n} + 2\pi m \right|.$$

Since

$$0 \le \varepsilon_n(\theta) \le \frac{\pi}{n} < \frac{\pi}{2},$$

the ray of angle θ meets one of the sides whose outward normal realizes $\varepsilon_n(\theta)$ (at a ray through a vertex, the two adjacent sides both do), at the point $\rho e^{i\theta}$. Substituting

$$x = \rho\cos(\theta), \qquad y = \rho\sin(\theta)$$

into the equation of that side gives

$$\rho\cos\left(\theta - \frac{(2k+1)\pi}{n}\right) = \cos\left(\frac{\pi}{n}\right),$$

and since the angular difference on the left has absolute value $\varepsilon_n(\theta)$ (up to an irrelevant multiple of 2π), we obtain

$$\rho = \frac{\cos(\pi/n)}{\cos(\varepsilon_n(\theta))} = \rho_n(\theta).$$

Since Π_n is convex and contains the origin, the point $re^{i\theta}$ belongs to Π_n if and only if $r \le \rho = \rho_n(\theta)$, which proves the result. □

Corollary 3.6. *The minimal dimension of realization function for circulant stochastic matrices is given as follows:*

$$d_C(1) = 1, \qquad d_C(\lambda) = 2, \qquad (\lambda \in [-1, 1)),$$

and, for $\lambda = re^{i\theta} \notin [-1,1]$,

$$d_C(\lambda) = \min\left\{ n \geq 3 :\ r \leq \frac{\cos(\pi/n)}{\cos(\varepsilon_n(\theta))} \right\}$$

whenever the minimum exists, and $d_C(\lambda) = \infty$ *otherwise.*

Proof. The value $d_C(1) = 1$ is realized by the 1×1 matrix (1), and, for $\lambda \in [-1,1)$, the value 2 is realized because $\Pi_2 = [-1,1]$ while $\Pi_1 = \{1\}$. The remaining case is Theorem 3.5 combined with Theorem 3.1. □

The next proposition shows that the minimum in Corollary 3.6 exists for every point of the open unit disk, and bounds it explicitly.

Proposition 3.7. *For every* $\lambda \in \mathbb{C}$ *with* $|\lambda| < 1$,

$$d_C(\lambda) \leq \max\left\{ 3,\ \left\lceil \frac{\pi}{\arccos(|\lambda|)} \right\rceil \right\}.$$

In particular, d_C *is finite on the open unit disk.*

Proof. Write $r = |\lambda|$ and $\lambda = re^{i\theta}$. Let $n \geq 3$ be an integer satisfying

$$n \geq \frac{\pi}{\arccos(r)}.$$

Since the cosine is decreasing on $[0, \pi/2]$, we have

$$r \leq \cos\left(\frac{\pi}{n}\right).$$

The regular polygon Π_n is centered at the origin and contains the closed disk of radius $\cos(\pi/n)$. Hence $\lambda \in \Pi_n$. Equivalently, by Theorem 3.5,

$$r \leq \cos\left(\frac{\pi}{n}\right) \leq \rho_n(\theta).$$

Thus every integer $n \geq 3$ satisfying the displayed lower bound is admissible. The smallest such integer is

$$\max\left\{ 3, \left\lceil \frac{\pi}{\arccos(r)} \right\rceil \right\}.$$

□

Example 3.8. Let us determine $d_C(\lambda)$ for $\lambda = \frac{9}{10}i$, so that $r = \frac{9}{10}$ and $\theta = \frac{\pi}{2}$. The relevant angular distances are

$$\varepsilon_3\left(\frac{\pi}{2}\right) = \frac{\pi}{6}, \qquad \varepsilon_4\left(\frac{\pi}{2}\right) = \frac{\pi}{4}, \qquad \varepsilon_5\left(\frac{\pi}{2}\right) = \frac{\pi}{10}, \qquad \varepsilon_6\left(\frac{\pi}{2}\right) = 0.$$

Consequently,

$$\rho_3\left(\frac{\pi}{2}\right) = \frac{\cos(\pi/3)}{\cos(\pi/6)} = \frac{1}{\sqrt{3}} \approx 0.577, \qquad \rho_4\left(\frac{\pi}{2}\right) = \frac{\cos(\pi/4)}{\cos(\pi/4)} = 1,$$
$$\rho_5\left(\frac{\pi}{2}\right) = \frac{\cos(\pi/5)}{\cos(\pi/10)} \approx 0.851, \qquad \rho_6\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{6}\right) \approx 0.866.$$

The first two values show that λ lies outside Π_3 but belongs to Π_4. It follows from Corollary 3.6 that

$$d_C\left(\frac{9}{10}i\right) = 4.$$

This is easy to see geometrically: the point i is a vertex of Π_4, so the segment joining 0 to i lies in Π_4 and contains λ.

It is worth observing that increasing the dimension does not necessarily preserve membership in the corresponding eigenvalue region. Indeed, the values computed above show that λ belongs to neither Π_5 nor Π_6. On the other hand,

$$\rho_7\Big(\frac{\pi}{2}\Big) = \frac{\cos(\pi/7)}{\cos(\pi/14)} \approx 0.924 > \frac{9}{10},$$

so $\lambda \in \Pi_7$. Thus λ belongs to Π_4 and Π_7, but not to the intervening regions Π_5 and Π_6, illustrating the nonnestedness discussed in Remark 3.2.

Finally, Proposition 3.7 yields the general estimate

$$d_C(\lambda) \le \left\lceil \frac{\pi}{\arccos(9/10)} \right\rceil = 7.$$

The estimate is not sharp here because it uses the disk inscribed in Π_n and therefore does not take advantage of the greater distance from the origin to a vertex. In dimension four, that distinction allows λ to be realized already.

Remark 3.9. One has

$$\bigcup_{n\ge1}\Pi_n = \{z \in \mathbb{C} : \ |z| < 1\} \cup \{\zeta \in \mathbb{C} : \ \zeta \text{ is a root of unity}\}, \qquad \text{so that} \qquad \overline{\bigcup_{n\ge1}\Pi_n} = \overline{\mathbb{D}}.$$

Indeed, the inradius of Π_n is $\cos(\pi/n)$, which tends to 1 as $n \to \infty$, so every point of the open unit disk belongs to Π_n for all sufficiently large n; on the other hand, the only points of Π_n lying on the unit circle are its vertices, namely the n-th roots of unity. Consequently, $d_C(\lambda) = \infty$ for every point λ of the unit circle that is not a root of unity, while, for a primitive m-th root of unity ζ, the polygons containing ζ are exactly those Π_n with $m \mid n$, whence

$$d_C(\zeta) = m = \operatorname{ord}(\zeta),$$

in accordance with the value obtained for the permutation class in Section 2.

Remark 3.10. Every circulant matrix has constant column sums as well as constant row sums, so every circulant stochastic matrix is doubly stochastic. Consequently, denoting by Ω the class of doubly stochastic matrices,

$$\Theta_n(C) \subseteq \mathrm{DS}_n \qquad \text{and} \qquad \bigcup_{k\le n}\Theta_k(C) = \bigcup_{k\le n}\Pi_k = \mathrm{PM}_n,$$

the Perfect–Mirsky region discussed in the introduction. Two consequences deserve mention. First,

$$d_\Omega(\lambda) \le d_C(\lambda), \qquad (\lambda \in \mathbb{C}),$$

so the results of this subsection furnish explicit upper bounds for the doubly stochastic realization function. Second, in every dimension for which the Perfect–Mirsky equality $\mathrm{DS}_n = \mathrm{PM}_n$ is known to hold (all $n \le 4$; see [13,14]), membership of a point in DS_n is decided precisely by the polygonal geometry developed above. The realization problem for the class Ω itself remains open, as noted in Section 2. In Subsection 4.2, the regions PM_n will themselves be realized as the spectral regions of a class of stochastic matrices.

3.2 Anticirculant stochastic matrices

We now consider the class $\mathcal{A}$ of anticirculant stochastic matrices. Beyond their intrinsic interest as the symmetric counterparts of circulant matrices, their study is motivated by the fact, established at the end of this subsection, that they coincide exactly with the Hankel stochastic matrices. Whereas circulant stochastic matrices may possess nonreal eigenvalues, anticirculant stochastic matrices have entirely real spectra and admit a complete spectral description.

Recall that an $n \times n$ matrix $A = (a_{ij})$, with row and column indices ranging over $\{0, 1, \dots, n-1\}$, is called *anticirculant* if there exist numbers $a_0, a_1, \dots, a_{n-1}$ such that

$$a_{ij} = a_{(i+j) \bmod n}, \qquad (0 \le i, j \le n-1).$$

Equivalently,

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_1 & a_2 & a_3 & \cdots & a_0 \\ a_2 & a_3 & a_4 & \cdots & a_1 \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n-1} & a_0 & a_1 & \cdots & a_{n-2} \end{pmatrix}.$$

Observe that every anticirculant matrix is symmetric; consequently all of its eigenvalues are real.

Throughout this subsection we assume that A is stochastic. Thus

$$a_r \ge 0, \qquad (0 \le r \le n-1), \qquad \sum_{r=0}^{n-1} a_r = 1.$$

To determine the spectrum of A, let

$$\omega = \mathrm{e}^{2\pi \mathrm{i}/n},$$

and define the Fourier vectors

$$v_k = (1, \omega^k, \omega^{2k}, \dots, \omega^{(n-1)k})^{\intercal}, \qquad (k = 0, \dots, n-1),$$

the index k being understood modulo n, so that $v_{-k} = v_{n-k}$. For circulant matrices, each vector v_k is an eigenvector. In the anticirculant case, the matrix instead sends v_k to a scalar multiple of v_{-k}. Indeed, for $0 \le i \le n-1$, the substitution $r = (i + j) \bmod n$ gives

$$(Av_k)_i = \sum_{j=0}^{n-1} a_{(i+j) \bmod n}\, \omega^{kj} = \sum_{r=0}^{n-1} a_r\, \omega^{k(r-i)} = \widehat{a}_k\, \omega^{-ki}, \qquad \text{where} \qquad \widehat{a}_k := \sum_{r=0}^{n-1} a_r\, \omega^{kr}.$$

Since the i-th component of v_{-k} is ω^{-ki}, this means

$$Av_k = \widehat{a}_k\, v_{-k} \qquad \text{and, interchanging the roles of } k \text{ and } -k, \qquad Av_{-k} = \widehat{a}_{-k}\, v_k.$$

Thus A leaves invariant the subspace

$$E_k := \operatorname{span}\{v_k, v_{-k}\}.$$

Suppose first that $k \not\equiv -k \pmod n$. Then v_k and v_{-k} are distinct Fourier vectors and are therefore linearly independent, and relative to the basis (v_k, v_{-k}) of E_k the action of A is represented by

$$\begin{pmatrix} 0 & \widehat{a}_{-k} \\ \widehat{a}_k & 0 \end{pmatrix},$$

whose characteristic polynomial is $\lambda^2 - \widehat{a}_k \widehat{a}_{-k}$. Since the coefficients a_r are real, $\widehat{a}_{-k} = \overline{\widehat{a}_k}$, so $\widehat{a}_k \widehat{a}_{-k} = \left|\widehat{a}_k\right|^2$ and the corresponding eigenvalues are $\pm\left|\widehat{a}_k\right|$. The exceptional cases are those with $k \equiv -k \pmod n$, namely $k = 0$ and, when n is even, $k = n/2$; there $Av_k = \widehat{a}_k v_k$, so $\widehat{a}_k$ itself is an eigenvalue. One has

$$\widehat{a}_0 = \sum_{r=0}^{n-1} a_r = 1, \qquad \text{and, for even } n, \qquad \widehat{a}_{n/2} = \sum_{r=0}^{n-1} (-1)^r a_r \in \mathbb{R},$$

while, for every k,

$$\left|\widehat{a}_k\right| = \left|\sum_{r=0}^{n-1} a_r \omega^{kr}\right| \leq \sum_{r=0}^{n-1} a_r = 1.$$

Finally, since the Fourier vectors $v_0, \dots, v_{n-1}$ form a basis of $\mathbb{C}^n$, the subspaces E_k exhaust $\mathbb{C}^n$, so the eigenvalues listed above account for the entire spectrum. We have proved the following theorem.

Theorem 3.11. *Let A be an $n \times n$ anticirculant stochastic matrix.*

1. *If n is odd, then*
$$\sigma(A) = \left\{1, \pm\left|\widehat{a}_1\right|, \pm\left|\widehat{a}_2\right|, \dots, \pm\left|\widehat{a}_{(n-1)/2}\right|\right\}.$$
2. *If n is even, then*
$$\sigma(A) = \left\{1, \pm\left|\widehat{a}_1\right|, \dots, \pm\left|\widehat{a}_{n/2-1}\right|, \widehat{a}_{n/2}\right\}.$$
3. *Every eigenvalue of A belongs to the interval $[-1, 1]$.*

The preceding theorem immediately determines the eigenvalue region for anticirculant stochastic matrices.

Corollary 3.12. *For every $n \geq 2$,*

$$\Theta_n(\mathcal{A}) = [-1, 1].$$

Proof. By Theorem 3.11, every eigenvalue of an anticirculant stochastic matrix belongs to $[-1, 1]$.

Conversely, fix $n \geq 2$ and let $\lambda \in [-1, 1]$. For $n = 2$, the matrix

$$A_\lambda := \begin{pmatrix} a & 1-a \\ 1-a & a \end{pmatrix}, \qquad a = \frac{1+\lambda}{2},$$

is anticirculant, stochastic, and has eigenvalues 1 and λ.

Now assume $n \geq 3$. Put $r = |\lambda|$ and define

$$a_0 = r + \frac{1-r}{n}, \qquad a_j = \frac{1-r}{n}, \qquad (j = 1, \dots, n-1).$$

Then $a_j \geq 0$ and $\sum_{j=0}^{n-1} a_j = 1$. Moreover, for $\omega = \mathrm{e}^{2\pi \mathrm{i}/n}$,

$$\begin{aligned} \widehat{a}_1 &= \sum_{j=0}^{n-1} a_j \omega^j \\ &= r + \frac{1-r}{n} \sum_{j=0}^{n-1} \omega^j \\ &= r. \end{aligned}$$

Since $n \geq 3$, we have $1 \not\equiv -1 \pmod{n}$, so Theorem 3.11 shows that the corresponding anticirculant stochastic matrix has both r and $-r$ as eigenvalues. Hence it realizes λ. □

We may therefore compute the realization dimension explicitly.

Corollary 3.13. *The minimal dimension of realization function associated to the class of anticirculant stochastic matrices is given by*

$$d_{\mathcal{A}}(1) = 1, \qquad d_{\mathcal{A}}(\lambda) = 2, \qquad (\lambda \in [-1, 1)),$$

while

$$d_{\mathcal{A}}(\lambda) = \infty, \qquad (\lambda \in \overline{\mathbb{D}} \setminus [-1,1]).$$

Proof. Let $\lambda \in [-1,1)$. By Corollary 3.12, λ is realized by a 2×2 anticirculant stochastic matrix, so $d_{\mathcal{A}}(\lambda) \leq 2$; since the only 1×1 stochastic matrix is (1) and $\lambda \neq 1$, in fact $d_{\mathcal{A}}(\lambda) = 2$. Finally, Theorem 3.11 shows that all eigenvalues of anticirculant stochastic matrices are real, so no nonreal number is realizable. □

To conclude this section, we show that the analysis above also disposes of the class of Hankel stochastic matrices. Recall that an $n \times n$ matrix $H = (H_{ij})$, with row and column indices ranging over $\{0, 1, \dots, n-1\}$, is *Hankel* if its entries are constant along anti-diagonals; equivalently, if there exist numbers $h_0, h_1, \dots, h_{2n-2}$ such that

$$H_{ij} = h_{i+j}, \qquad (0 \leq i, j \leq n-1).$$

Every Hankel matrix is determined by its anti-diagonals; for stochastic Hankel matrices, the row-sum condition imposes strong compatibility relations among these anti-diagonals, which turn out to force precisely the periodicity required for anticirculant structure.

Proposition 3.14. *For every $n \geq 1$, the class of $n \times n$ Hankel stochastic matrices coincides with the class of $n \times n$ anticirculant stochastic matrices.*

Proof. Every anticirculant matrix is Hankel. Indeed, if $A = (a_{(i+j) \bmod n})_{0 \leq i,j \leq n-1}$ is anticirculant, define

$$h_r := a_{r \bmod n}, \qquad (r = 0, \dots, 2n-2).$$

Then $A_{ij} = h_{i+j}$ for all i, j, so A is Hankel.

Conversely, let $H = (h_{i+j})_{0 \leq i,j \leq n-1}$ be a Hankel stochastic matrix. Since H is stochastic,

$$\sum_{j=0}^{n-1} h_{i+j} = 1, \qquad (i = 0, \dots, n-1).$$

For $i = 0, \dots, n-2$, subtracting the equation corresponding to i from the equation corresponding to $i+1$ gives

$$h_{i+n} - h_i = 0,$$

that is,

$$h_i = h_{i+n}, \qquad (i = 0, \dots, n-2).$$

Now define $a_r := h_r$ for $r = 0, \dots, n-1$. If $0 \leq i + j \leq n-1$, then $H_{ij} = h_{i+j} = a_{i+j}$; if $n \leq i + j \leq 2n-2$, then $H_{ij} = h_{i+j} = h_{i+j-n} = a_{i+j-n}$. Hence, in all cases, $H_{ij} = a_{(i+j) \bmod n}$, and H is anticirculant. □

Corollary 3.15. *Let $\mathcal{H}$ denote the class of Hankel stochastic matrices. Then $\Theta_n(\mathcal{H}) = \Theta_n(\mathcal{A})$ for every $n \geq 1$; in particular, $\Theta_n(\mathcal{H}) = [-1,1]$ for every $n \geq 2$, no nonreal number is an eigenvalue of a Hankel stochastic matrix, and $d_{\mathcal{H}} = d_{\mathcal{A}}$.*

Proof. By Proposition 3.14, the two classes coincide; the conclusions follow from Corollaries 3.12 and 3.13. □

Thus, unlike the Toeplitz condition, the Hankel condition does not lead back to the circulant hierarchy: stochastic Hankel matrices inherit the trivial realization theory of the anticirculant class.

4 Classes generated by a single permutation

The classes studied in Section 3 were defined by symmetry conditions on the entries. In this section we consider two classes defined dynamically from a single permutation. In the first, at each step the chain performs the permutation move with a fixed probability and stays put otherwise; in the second, it performs the move a random number of times per step. The first class leads to a star of chords and an arithmetic realization function; the second leads to the Perfect–Mirsky regions themselves. Both classes consist of doubly stochastic matrices, and the second contains the circulant class of Section 3; together they connect the circulant theory developed there to the doubly stochastic single eigenvalue problem recalled in the introduction.

4.1 Lazy permutation chains

A standard device in the theory of Markov chains (see, e.g., [35, Section 1.3]) is to replace a transition matrix Q by its *lazy* version

$$\frac{1}{2}(I+Q),$$

so that at each step the chain either stays at its current state or follows the transition prescribed by Q, each with probability $1/2$. More generally, one may consider the one-parameter family

$$(1-t)I+tQ, \qquad t\in[0,1].$$

We apply this construction to the deterministic dynamics defined by a permutation. Given an $n\times n$ permutation matrix P and $t\in[0,1]$, let

$$L=(1-t)I_n+tP.$$

Then L is stochastic: at each step, the chain applies the permutation with probability t and remains at its current state with probability $1-t$. By analogy with the standard lazy construction, we denote by $\mathcal{L}$ the class of all such matrices and refer to the corresponding chains as *lazy permutation chains*.

Beyond this probabilistic interpretation, $\mathcal{L}$ is precisely the subclass of the convex combinations of two permutation matrices, discussed in Section 2, in which one of the two permutations is the identity.

For $\zeta\in\mathbb{C}$ with $|\zeta|=1$, let

$$[1,\zeta]:=\{(1-t)+t\zeta:\ t\in[0,1]\}$$

denote the chord of the closed unit disk joining 1 to ζ.

Theorem 4.1. *For every $n\geq 1$,*

$$\Theta_n(\mathcal{L})=\bigcup\{[1,\zeta]:\ \zeta \text{ is a root of unity with } \operatorname{ord}(\zeta)\leq n\}.$$

In particular, the regions $\Theta_n(\mathcal{L})$ form a nested sequence.

Proof. Since $(1-t)I_n+tP$ is a polynomial in P, its eigenvalues are

$$\sigma\big((1-t)I_n+tP\big)=\{(1-t)+t\mu:\ \mu\in\sigma(P)\}.$$

As recalled in Section 2, the eigenvalues of the $n\times n$ permutation matrices are exactly the roots of unity of order at most n. Hence every eigenvalue of a matrix of order n in $\mathcal{L}$ lies on a chord $[1,\ \mu]$ with $\operatorname{ord}(\mu)\leq n$. Conversely, let ζ be a root of unity with $m:=\operatorname{ord}(\zeta)\leq n$ and let $t\in[0,1]$. If P is the permutation matrix of an m-cycle completed by $n-m$ fixed points, then $\zeta\in\sigma(P)$, so $(1-t)+t\zeta$ is an eigenvalue of $(1-t)I_n+tP\in\mathcal{L}$. This proves the displayed identity. The nestedness is clear from it, and also follows from the fact that $\mathcal{L}$ is closed under the operation $S\mapsto S\oplus(1)$ of Section 2. □

Thus $\Theta_n(\mathcal{L})$ is a *star of chords* issuing from the common point 1: the chords of the closed unit disk joining 1 to the roots of unity of order at most n. The associated realization problem is solved by identifying, for a prescribed eigenvalue, the unique chord through 1 on which it lies.

Theorem 4.2. *The minimal dimension of realization function associated with the class $\mathcal{L}$ is given as follows. First, $d_{\mathcal{L}}(1) = 1$. Next, for $\lambda \in \overline{\mathbb{D}} \setminus \{1\}$, set*

$$\zeta_\lambda := -\frac{1-\lambda}{\overline{1-\lambda}},$$

which is the second point at which the chord of $\overline{\mathbb{D}}$ through 1 and λ meets the unit circle. Then

$$d_{\mathcal{L}}(\lambda) = \begin{cases} \operatorname{ord}(\zeta_\lambda) & \text{if } \zeta_\lambda \text{ is a root of unity,} \\ \infty & \text{otherwise.} \end{cases}$$

Proof. Let $\lambda \in \overline{\mathbb{D}} \setminus \{1\}$ and write $w := 1-\lambda$. Since $|\lambda| \le 1$ and $\lambda \neq 1$, we have $\operatorname{Re}(w) = 1 - \operatorname{Re}(\lambda) > 0$; in particular $w \neq 0$ and ζ_λ is well defined, with $|\zeta_\lambda| = 1$.

We first verify the geometric description of ζ_λ. The points of the line through 1 and λ are

$$z_s = 1 + s(\lambda - 1) = 1 - sw, \qquad (s \in \mathbb{R}),$$

and

$$|z_s|^2 = 1 - 2s\operatorname{Re}(w) + s^2|w|^2,$$

so $|z_s| = 1$ precisely for $s = 0$ and for

$$s_* := \frac{2\operatorname{Re}(w)}{|w|^2} > 0.$$

Using $|w|^2 = w\overline{w}$ and $2\operatorname{Re}(w) = w + \overline{w}$, the second intersection of the line with the unit circle is

$$z_{s_*} = 1 - \frac{w+\overline{w}}{\overline{w}} = -\frac{w}{\overline{w}} = \zeta_\lambda.$$

Moreover, $\lambda = z_1$ and $|z_1| = |\lambda| \le 1$; since the quadratic $s \mapsto |z_s|^2 - 1$ is nonpositive exactly on $[0, s_*]$, it follows that $1 \le s_*$, that is, λ lies on the chord $[1, \zeta_\lambda]$.

Now let $n \ge 1$. By Theorem 4.1, $\lambda \in \Theta_n(\mathcal{L})$ if and only if $\lambda \in [1, \zeta]$ for some root of unity ζ with $\operatorname{ord}(\zeta) \le n$. If $\lambda \in [1, \zeta]$ for some ζ on the unit circle, then, since $\lambda \neq 1$, the line through 1 and λ coincides with the line through 1 and ζ, and hence ζ is its second intersection with the unit circle, namely $\zeta = \zeta_\lambda$. The only chord through 1 containing λ is therefore $[1, \zeta_\lambda]$, and consequently

$$\lambda \in \Theta_n(\mathcal{L}) \qquad \Longleftrightarrow \qquad \zeta_\lambda \text{ is a root of unity with } \operatorname{ord}(\zeta_\lambda) \le n.$$

The claimed formula for $d_{\mathcal{L}}$ follows. □

The formula of Theorem 4.2 admits a purely arithmetic form. For a nonzero complex number w, we denote by $\operatorname{Arg}(w) \in (-\pi, \pi]$ its principal argument.

Corollary 4.3. *Let $\lambda \in \overline{\mathbb{D}} \setminus \{1\}$ and put $\varphi := \operatorname{Arg}(1-\lambda)$. Then $\varphi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and*

$$d_{\mathcal{L}}(\lambda) < \infty \qquad \Longleftrightarrow \qquad \frac{\varphi}{\pi} \in \mathbb{Q}.$$

In that case, write $\varphi/\pi = a/b$ in lowest terms, with $a \in \mathbb{Z}$ and $b \geq 1$ (so that $a = 0$ and $b = 1$ when $\varphi = 0$). Then

$$d_{\mathcal{L}}(\lambda) = \frac{2b}{\gcd(2b,\ b+2a)} = \begin{cases} 2b & \text{if } b \text{ is odd,} \\ b & \text{if } b \equiv 0 \pmod 4, \\ b/2 & \text{if } b \equiv 2 \pmod 4. \end{cases}$$

Proof. Step 1: the expression of ζ_λ. Write $w := 1 - \lambda$. It was shown in the proof of Theorem 4.2 that $\operatorname{Re}(w) > 0$; hence $\varphi = \operatorname{Arg}(w) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Since $w = |w|e^{i\varphi}$ and $\overline{w} = |w|e^{-i\varphi}$,

$$\zeta_\lambda = -\frac{w}{\overline{w}} = -e^{2i\varphi} = e^{i(\pi+2\varphi)}.$$

Step 2: finiteness. A complex number $e^{i\psi}$, $\psi \in \mathbb{R}$, is a root of unity if and only if $\psi/(2\pi) \in \mathbb{Q}$. For $\psi = \pi + 2\varphi$ we have $\psi/(2\pi) = \frac{1}{2} + \varphi/\pi$, so ζ_λ is a root of unity if and only if $\varphi/\pi \in \mathbb{Q}$. The first assertion now follows from Theorem 4.2.

Step 3: the order of ζ_λ. Assume that $\varphi/\pi = a/b$ in lowest terms. By Step 1,

$$\zeta_\lambda = \exp\left(\frac{2\pi i\,(b+2a)}{2b}\right).$$

For integers $N \geq 1$ and k, the order of $e^{2\pi i k/N}$ is $N/\gcd(N,k)$. Applying this with $N = 2b$ and $k = b + 2a$, Theorem 4.2 gives $d_{\mathcal{L}}(\lambda) = \operatorname{ord}(\zeta_\lambda) = 2b/\gcd(2b, b+2a)$.

Step 4: evaluation of the greatest common divisor. Put $g := \gcd(2b, b+2a)$ and recall that $\gcd(a,b) = 1$, so that $\gcd(a,c) = 1$ for every divisor c of b. If b is odd, then $b + 2a$ is odd, so $g = \gcd(b, b+2a) = \gcd(b, 2a) = \gcd(b, a) = 1$. Suppose now that b is even; then a is odd, and we write $b = 2c$, so that $g = 2\gcd(2c, c+a)$. If c is even, then $c + a$ is odd, whence $\gcd(2c, c+a) = \gcd(c, c+a) = \gcd(c, a) = 1$ and $g = 2$. If c is odd, then $c + a$ is even and $\gcd(2c, c+a) = 2\gcd\left(c, \frac{c+a}{2}\right)$; any positive common divisor of c and $\frac{c+a}{2}$ divides both c and $c + a$, hence divides $\gcd(c, a) = 1$, so that $\gcd(2c, c+a) = 2$ and $g = 4$. The three cases give $d_{\mathcal{L}}(\lambda) = 2b$, b and $b/2$, respectively. □

Example 4.4. We illustrate Theorem 4.2 and Corollary 4.3 in three cases.

1. Let $\lambda \in [-1, 1)$ be real. Then $\varphi = 0$, so $a = 0$, $b = 1$ and $d_{\mathcal{L}}(\lambda) = 2$. Indeed, $\zeta_\lambda = -1$, and the chord joining 1 to -1 is the interval $[-1, 1]$, which consists of the eigenvalues of the matrices $(1-t)I_2 + tP$, where P is the 2×2 transposition matrix.
2. Let $\lambda \neq 1$ be a point of the unit circle. Then $\overline{\lambda} = 1/\lambda$, and a direct computation gives $\zeta_\lambda = \lambda$. Hence $d_{\mathcal{L}}(\lambda) = \operatorname{ord}(\lambda)$ if λ is a root of unity and $d_{\mathcal{L}}(\lambda) = \infty$ otherwise, in agreement with the values of $d_{\mathcal{P}}$ obtained for the permutation class in Section 2.
3. Let $\lambda = \frac{1+i}{2}$. Then $1 - \lambda = \frac{1-i}{2}$ and $\varphi = -\frac{\pi}{4}$, so $a = -1$ and $b = 4$. Since $b \equiv 0 \pmod 4$, we get $d_{\mathcal{L}}(\lambda) = 4$; equivalently, $\zeta_\lambda = -e^{-i\pi/2} = i$ has order 4. The point λ is the midpoint of the chord $[1, i]$, and it is first realized as an eigenvalue of $\frac{1}{2}(I_4 + P)$ with P a 4-cycle.

Remark 4.5. By Theorem 4.1,

$$\Theta(\mathcal{L}) = \bigcup_{\zeta \text{ a root of unity}} [1, \zeta].$$

This set is dense in $\overline{\mathbb{D}}$: every point $\lambda \in \overline{\mathbb{D}}$ lies on some chord through 1, say $\lambda = (1-t) + t\eta$ with $0 \leq t \leq 1$ and $|\eta| = 1$, and if roots of unity ζ_j tend to η, then the points $(1-t) + t\zeta_j$ of $\Theta(\mathcal{L})$ tend to λ. Being a countable union of line segments, however, $\Theta(\mathcal{L})$ has planar Lebesgue measure zero, so that $d_{\mathcal{L}}$ is finite on a dense set but infinite almost everywhere.

Moreover, $d_{\mathcal{L}}$ is unbounded on every nonempty relatively open subset U of $\overline{\mathbb{D}}$, even when attention is restricted to the points of U at which it is finite. To see this, pick a point $\lambda \in U$ in the open disk, write

$\lambda = (1-t) + t\eta$ with $0 < t < 1$ and $|\eta| = 1$, and choose primitive roots of unity $\zeta_j \to \eta$ whose orders tend to infinity (roots of increasing prime orders will do). The points $\lambda_j := (1-t) + t\zeta_j$ then belong to U for all sufficiently large j, while Theorem 4.2 gives $d_{\mathcal{L}}(\lambda_j) = \operatorname{ord}(\zeta_j) \to \infty$. This behavior contrasts with that of the circulant class, for which Proposition 3.7 shows that d_C is finite and locally bounded on the open disk.

Finally, since distinct chords $[1, \zeta]$ meet only at 1, Theorem 4.2 identifies, for every $m \geq 2$, the level set $\{\lambda : d_{\mathcal{L}}(\lambda) = m\}$ with the union of the punctured chords $[1, \zeta] \setminus \{1\}$ over the primitive m-th roots of unity ζ.

Since $\mathcal{L} \subseteq \Sigma$, the function $d_{\mathcal{L}}$ also furnishes an upper bound for d_Σ wherever it is finite. The argument of this subsection does not, however, extend to convex combinations of two arbitrary permutations: although one may write

$$(1-t)Q + tP = Q\big((1-t)I + tQ^{-1}P\big),$$

the two factors need not commute, and the spectral mapping argument used above says nothing about the spectrum of their product.

4.2 Convex combinations of powers of a permutation

There is, however, a different direction in which the class $\mathcal{L}$ can be enlarged while remaining completely solvable: instead of admitting a second arbitrary permutation, one may admit all the powers of the first. Given an $n \times n$ permutation matrix P and a probability vector $(a_0, a_1, a_2, \dots)$ with finitely many nonzero entries, the matrix

$$G = \sum_{k \geq 0} a_k P^k$$

is stochastic; indeed doubly stochastic, being a convex combination of permutation matrices. Probabilistically, the associated chain applies the deterministic map underlying P a *random* number of times per unit of time, the number being drawn according to the distribution (a_k); the lazy chains of the preceding subsection correspond to distributions supported on $\{0, 1\}$. We denote by $\mathcal{G}$ the class of all such matrices. Equivalently, the matrices of order n in $\mathcal{G}$ are the elements of the convex hulls $\operatorname{Co}(\langle P \rangle)$ of the cyclic groups generated by the $n \times n$ permutation matrices; negative powers contribute nothing new, since P has finite order.

The class $\mathcal{G}$ contains $\mathcal{L}$ (take $a_k = 0$ for $k \geq 2$), and it also contains the circulant class C of Section 3: every circulant stochastic matrix is of the form $\sum_k a_k C^k$, where $C = \operatorname{Circ}(0, 1, 0, \dots, 0)$ is the cyclic shift. Both inclusions are strict. For instance, if P is the permutation matrix of a 3-cycle completed by one fixed point, then the matrix $\frac{1}{3}\big(I_4 + P + P^2\big) \in \mathcal{G}$ is not of the form $(1-t)I_4 + tQ$ (its diagonal entry $\frac{1}{3}$ forces $t = \frac{2}{3}$, whereas its nonzero off-diagonal entries equal $\frac{1}{3} \neq t$), nor is it circulant (its diagonal is not constant).

The spectral regions of $\mathcal{G}$ turn out to be exactly the Perfect–Mirsky regions encountered in the introduction and in Remark 3.10. This identification is a special case of the classification of the hull spectra of abelian matrix groups obtained by Jankowski, Johnson and Lim [18]; since the argument is short in the present situation, we include it for completeness.

Theorem 4.6 (see [18, Corollary 3.5]). *For every $n \geq 1$,*

$$\Theta_n(\mathcal{G}) = \bigcup_{m \leq n} \Pi_m = \mathrm{PM}_n.$$

In particular, the regions $\Theta_n(\mathcal{G})$ form a nested sequence.

Proof. Let $G = \sum_{k \geq 0} a_k P^k \in \mathcal{G}$, where P is an $n \times n$ permutation matrix. Since G is a polynomial in P, its eigenvalues are the numbers $\sum_{k \geq 0} a_k \mu^k$ with $\mu \in \sigma(P)$. Fix such a μ and let $m := \operatorname{ord}(\mu) \leq n$. Since μ^k depends only on the residue of k modulo m, grouping the coefficients accordingly gives

$$\sum_{k \geq 0} a_k \mu^k = \sum_{r=0}^{m-1} b_r \, \mu^r, \qquad b_r := \sum_{k \equiv r \pmod m} a_k,$$

where $(b_0, \dots, b_{m-1})$ is again a probability vector. As μ is a primitive m-th root of unity, its powers $\mu^0,\ \mu^1, \dots,\ \mu^{m-1}$ are precisely the m-th roots of unity, so the displayed quantity is a convex combination of the m-th roots of unity, that is, a point of $\Pi_m \subseteq \mathrm{PM}_n$. Hence $\Theta_n(\mathcal{G}) \subseteq \mathrm{PM}_n$.

Conversely, let $\lambda \in \Pi_m$ with $m \le n$, say

$$\lambda = \sum_{r=0}^{m-1} b_r \, \mathrm{e}^{2\pi \mathrm{i} r/m}$$

with $(b_0, \dots, b_{m-1})$ a probability vector, and let P be the permutation matrix of an m-cycle completed by $n-m$ fixed points. Then $\mu := \mathrm{e}^{2\pi \mathrm{i}/m} \in \sigma(P)$, and if v is a corresponding eigenvector, then

$$\left(\sum_{r=0}^{m-1} b_r P^r\right) v = \left(\sum_{r=0}^{m-1} b_r \mu^r\right) v = \lambda v,$$

so λ is an eigenvalue of the matrix $\sum_r b_r P^r \in \mathcal{G}$. Hence $\mathrm{PM}_n \subseteq \Theta_n(\mathcal{G})$. Finally, the regions PM_n are nested by their very definition. □

Corollary 4.7. *The classes $\mathcal{G}$ and $\mathcal{C}$ have the same minimal dimension of realization function:*

$$d_{\mathcal{G}} = d_{\mathcal{C}}.$$

In particular, $d_{\mathcal{G}}$ is computed by the radial description of Corollary 3.6, obeys the upper bound of Proposition 3.7, and is finite on the entire open unit disk.

Proof. Both functions are infinite off $\bigcup_{m\ge1} \Pi_m$, and on this union both equal $\min\{m \ge 1 : \lambda \in \Pi_m\}$. For $d_{\mathcal{C}}$ this is the content of Theorem 3.1; for $d_{\mathcal{G}}$ it follows from Theorem 4.6, since $\lambda \in \mathrm{PM}_n$ for some n if and only if $\lambda \in \Pi_m$ for some $m \le n$, and the smallest such n equals the smallest such m. □

Remark 4.8. Although $\mathcal{G}$ strictly contains $\mathcal{C}$, Corollary 4.7 shows that the two classes have the same realization function. Their fixed-dimensional regions are nevertheless different: for $n \ge 3$, PM_n strictly contains Π_n. Passing from $\mathcal{C}$ to $\mathcal{G}$ therefore makes the regions nested without reducing the first dimension in which any eigenvalue occurs.

The inclusions $\mathcal{L} \subseteq \mathcal{G} \subseteq \Omega$ and Corollary 4.7 give the comparisons

$$d_\Omega \le d_{\mathcal{G}} = d_{\mathcal{C}} \le d_{\mathcal{L}}.$$

Remark 4.9. Both the doubly stochastic hierarchy (DS_n) and the hierarchy (PM_n) are nested, so each is recovered from the associated realization function:

$$\mathrm{DS}_n = \{\lambda :\ d_\Omega(\lambda) \le n\} \qquad \text{and, by Theorem 4.6,} \qquad \mathrm{PM}_n = \left\{\lambda :\ d_{\mathcal{G}}(\lambda) \le n\right\}.$$

Consequently, the Perfect–Mirsky conjecture discussed in the introduction is *equivalent* to the equality

$$d_\Omega = d_{\mathcal{G}}$$

of the realization functions of the doubly stochastic class and of its subclass $\mathcal{G}$; informally, to the assertion that every eigenvalue of a doubly stochastic matrix is already an eigenvalue, in the same dimension, of a convex combination of powers of a single permutation matrix. By [13,14], the two functions agree at every λ with $d_\Omega(\lambda) \le 4$; by [15], they differ somewhere: at any point $\lambda_0 \in \mathrm{DS}_5 \setminus \mathrm{PM}_5$, one has $d_\Omega(\lambda_0) \le 5 < d_{\mathcal{G}}(\lambda_0)$.

5 Tridiagonal Toeplitz-type stochastic matrices

The classes studied so far were defined by global symmetry conditions. We now turn to the simplest class defined by a locality condition: stochastic matrices of bandwidth 1 whose interior rows are translates of one another. Recall that an $n \times n$ matrix $A = (a_{ij})$ is said to have *bandwidth* m if

$$a_{ij} = 0 \qquad \text{whenever} \qquad |i - j| > m;$$

thus tridiagonal matrices are precisely the matrices of bandwidth 1.

Concretely, we consider matrices of the form

$$T_n(p,q) := \begin{pmatrix} 1-p & p & 0 & \cdots & 0 \\ q & r & p & \ddots & \vdots \\ 0 & q & r & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & p \\ 0 & \cdots & 0 & q & 1-q \end{pmatrix},$$

where the parameters (p, q) range over the simplex

$$\Delta := \{(p,q) : \ p, q \geq 0, \ \ p + q \leq 1\}$$

and $r := 1 - p - q$. For $n = 1$ we set $T_1 = (1)$, and for $n = 2$,

$$T_2(p,q) = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}.$$

Observe that rows $2, \dots, n-1$ are translates of one another and therefore constitute a tridiagonal Toeplitz block with bands (q, r, p); the first and last rows are modified so that the matrix remains stochastic. We shall refer to these matrices as *tridiagonal Toeplitz-type stochastic matrices*, and we denote by $\mathcal{B}^{(1)}$ the class of all such matrices.

Remark 5.1. For $n \leq 2$ the matrix $T_n(p,q)$ contains no interior row, and $T_2(p,q)$ is stochastic for any $p, q \in [0,1]$, whether or not $p + q \leq 1$. We nevertheless insist that the parameters range over the simplex Δ for *every* n: the datum defining the class is the triple (q, r, p) of transition probabilities of the interior rows, and the requirement $r = 1 - p - q \geq 0$ is part of this datum. The convention matters. Were the constraint dropped for $n = 2$, the region $\Theta_2(\mathcal{B}^{(1)})$ determined below would become the full interval $[-1, 1]$ (the eigenvalues of $T_2(p,q)$ are 1 and $1 - p - q$), and the hierarchy of Theorem 5.3 would collapse at its second stage.

These matrices are precisely the transition matrices of the finite *birth–death chains*, or *nearest-neighbor random walks*, with constant transition probabilities: the states are the vertices $1, 2, \dots, n$ of a finite path, and from every interior state $i \in \{2, \dots, n-1\}$ the chain moves according to

$$P(i, i-1) = q, \qquad P(i,i) = r, \qquad P(i, i+1) = p,$$

while at the two endpoints the probabilities of the missing transitions are transferred to the diagonal, yielding the boundary rows $(1 - p, p, 0, \dots, 0)$ and $(0, \dots, 0, q, 1 - q)$. The realization problem for this class therefore asks a natural design question: how long must the path be for a prescribed spectral mode to occur? As was pointed out in the introduction, the homogeneity assumption is essential here: without it, every point of $[-1, 1]$ is an eigenvalue of some $n \times n$ tridiagonal doubly stochastic matrix as soon as $n \geq 2$, so that the regions coincide in every dimension, and it is the constancy of the transition probabilities that produces a nontrivial hierarchy.

For each $n \geq 1$, let

$$\Theta_n(\mathcal{B}^{(1)}) = \bigcup_{(p,q)\in\Delta} \sigma\big(T_n(p,q)\big)$$

denote the set of all eigenvalues that may occur for an $n \times n$ tridiagonal Toeplitz-type stochastic matrix. We determine these regions and the corresponding realization function.

The eigenvalues of the matrices $T_n(p,q)$ are known explicitly, as recalled in the introduction [20–23], and we briefly indicate where the formula comes from. For $p, q > 0$, a diagonal similarity replaces the two off-diagonal bands of $T_n(p,q)$ by the constant value $c := \sqrt{pq}$, and the characteristic polynomial of the resulting symmetric tridiagonal matrix factors as

$$\det\big(tI - T_n(p,q)\big) = (t-1)\, c^{n-1}\, U_{n-1}\Big(\frac{t-r}{2c}\Big),$$

where $r = 1 - p - q$ and U_{n-1} denotes the Chebyshev polynomial of the second kind. Since the zeros of U_{n-1} are the numbers $\cos(k\pi/n)$, $1 \leq k \leq n-1$, this yields the following statement.

Lemma 5.2 (see [20–23]). *For $p, q > 0$, the eigenvalues of $T_n(p,q)$ are*

$$\lambda_0 = 1$$

and

$$\lambda_k = 1 - p - q + 2\sqrt{pq}\cos\left(\frac{k\pi}{n}\right), \qquad (k = 1, \dots, n-1).$$

The same formulas remain valid when $p = 0$ or $q = 0$, by continuity of the eigenvalues.

Theorem 5.3. *For every $n \geq 1$,*

$$\Theta_n(\mathcal{B}^{(1)}) = \Big[-\cos\Big(\frac{\pi}{n}\Big),\, 1\Big].$$

Proof. For $n = 1$ the statement reads $\Theta_1(\mathcal{B}^{(1)}) = \{1\}$, which holds since $T_1 = (1)$; assume henceforth $n \geq 2$.

Set

$$s := p + q, \qquad u := 2\sqrt{pq}, \qquad c_k := \cos\left(\frac{k\pi}{n}\right), \quad (k = 1, \dots, n-1).$$

By the arithmetic-geometric mean inequality, $u \leq s$, and the stochasticity condition gives $s \leq 1$; thus

$$0 \leq u \leq s \leq 1.$$

By Lemma 5.2, every eigenvalue of $T_n(p,q)$ is either 1 or of the form

$$\lambda = 1 - s + u\, c_k$$

for some $k \in \{1, \dots, n-1\}$. We first show that every such λ lies in $[-\cos(\pi/n), 1]$. On the one hand, since $c_k \leq 1$,

$$\lambda \leq 1 - s + u \leq 1.$$

On the other hand, if $c_k \geq 0$, then $\lambda \geq 1 - s \geq 0 \geq -\cos(\pi/n)$, while if $c_k < 0$, then

$$\lambda = 1 - s + u\, c_k \geq 1 - s + s\, c_k = 1 - s(1 - c_k) \geq 1 - (1 - c_k) = c_k \geq c_{n-1} = -\cos\Big(\frac{\pi}{n}\Big).$$

Hence

$$\Theta_n(\mathcal{B}^{(1)}) \subseteq \Big[-\cos\Big(\frac{\pi}{n}\Big), 1\Big].$$

Conversely, consider the one-parameter family

$$p = q = \frac{s}{2}, \qquad s \in [0, 1].$$

For these matrices $u = s$, so Lemma 5.2 with $k = n-1$ exhibits the eigenvalue

$$\lambda_{n-1}(s) = 1 - s + s\,\cos\left(\frac{(n-1)\pi}{n}\right) = 1 - s\left(1 + \cos\left(\frac{\pi}{n}\right)\right),$$

which decreases continuously from 1 (at $s = 0$) to $-\cos(\pi/n)$ (at $s = 1$). Hence every point of $[-\cos(\pi/n), 1]$ belongs to $\Theta_n(\mathcal{B}^{(1)})$, which completes the proof. □

Remark 5.4. The one-parameter family used in the second half of the proof is symmetric, hence doubly stochastic, as is every symmetric stochastic matrix. Consequently, the doubly stochastic members of $\mathcal{B}^{(1)}$, the matrices $T_n(p, p)$ with $0 \le p \le \frac{1}{2}$, already realize the full region: this subclass has the same spectral regions and the same minimal dimension of realization function as $\mathcal{B}^{(1)}$ itself.

Thus the class of tridiagonal Toeplitz-type stochastic matrices gives rise to the nested hierarchy of intervals

$$\{1\} = \left[-\cos(\pi),\, 1\right] \subseteq \left[-\cos\left(\frac{\pi}{2}\right), 1\right] \subseteq \left[-\cos\left(\frac{\pi}{3}\right), 1\right] \subseteq \left[-\cos\left(\frac{\pi}{4}\right), 1\right] \subseteq \cdots,$$

whose union is the interval

$$\Theta(\mathcal{B}^{(1)}) = (-1, 1].$$

In particular, the hierarchy is nested, so the minimal dimension of realization function is a genuine threshold in the sense of Remark 2.2. It admits the following closed form.

Corollary 5.5. *The minimal dimension of realization function associated with the class of tridiagonal Toeplitz-type stochastic matrices is given by*

$$d_{\mathcal{B}^{(1)}}(\lambda) = \left\lceil \frac{\pi}{\arccos(-\lambda)} \right\rceil, \qquad (\lambda \in (-1, 1]),$$

while

$$d_{\mathcal{B}^{(1)}}(\lambda) = \infty, \qquad (\lambda \in \overline{\mathbb{D}} \setminus (-1, 1]).$$

Proof. Let $\lambda \in (-1, 1]$, so that $-\lambda \in [-1, 1)$ and $\arccos(-\lambda) \in (0, \pi]$. By Theorem 5.3, for every $n \ge 1$,

$$\lambda \in \Theta_n(\mathcal{B}^{(1)}) \qquad \Longleftrightarrow \qquad \lambda \ge -\cos\left(\frac{\pi}{n}\right) \qquad \Longleftrightarrow \qquad \cos\left(\frac{\pi}{n}\right) \ge -\lambda.$$

Since the cosine is decreasing on $[0, \pi]$ and both π/n and $\arccos(-\lambda)$ belong to $(0, \pi]$, the last condition is equivalent to

$$\frac{\pi}{n} \le \arccos(-\lambda), \qquad \text{that is,} \qquad n \ge \frac{\pi}{\arccos(-\lambda)}.$$

The smallest such positive integer is $\lceil \pi/\arccos(-\lambda) \rceil$. Finally, $\Theta(\mathcal{B}^{(1)}) = (-1, 1]$, so $d_{\mathcal{B}^{(1)}}(\lambda) = \infty$ off this interval. □

Remark 5.6. The formula in Corollary 5.5 shows that the realization dimension is 1 only at $\lambda = 1$, and is 2 for $\lambda \in [0, 1)$. For $-1 < \lambda < 0$, we have

$$d_{\mathcal{B}^{(1)}}(\lambda) = \left\lceil \frac{\pi}{\arccos(-\lambda)} \right\rceil,$$

which tends to infinity as λ approaches -1. In fact, using the asymptotic relation

$$\arccos(1 - \varepsilon) \sim \sqrt{2\varepsilon} \qquad (\varepsilon \to 0^+),$$

we obtain

$$d_{\mathcal{B}^{(1)}}(\lambda) \sim \frac{\pi}{\sqrt{2(1+\lambda)}} \qquad (\lambda \to -1^+).$$

Thus, although every real number strictly between -1 and 1 can be realized in this class, realizing eigenvalues increasingly close to -1 requires matrices of increasingly large dimension.

The same expression involving arccos appears in the circulant upper bound of Proposition 3.7. In both cases, the relevant quantity is $\cos(\pi/n)$. For circulant matrices, it is the radius of the disk inscribed in the regular polygon Π_n; for tridiagonal matrices, its negative gives the left endpoint of the eigenvalue interval $\Theta_n(\mathcal{B}^{(1)})$.

6 Pentadiagonal Toeplitz-type stochastic matrices: low-dimensional exact results

We now allow jumps of length two while retaining the spatial homogeneity and the boundary convention of the tridiagonal model. The change is more significant than it may appear: the resulting pentadiagonal matrices need no longer be similar to symmetric ones, and they may have nonreal eigenvalues, so that the realization problem leaves the real axis. We do not attempt a description of the eigenvalue regions in every dimension. Instead, we determine their real slices through dimension four, compute the largest absolute imaginary part in dimensions three and four, and use these bounds to exhibit an explicit nonreal eigenvalue whose minimal realization dimension is five. As explained in the introduction, the available results on pentadiagonal Toeplitz matrices concern the symmetric case and do not apply directly to the matrices considered here.

6.1 The class and a star-shaped reduction

Let

$$p_{-2},\ p_{-1},\ p_0,\ p_1,\ p_2 \geq 0, \qquad p_{-2}+p_{-1}+p_0+p_1+p_2 = 1.$$

For $-2 \leq k \leq 2$, let $Q_k^{(n)}$ denote the $n \times n$ 0-1 stochastic matrix of the deterministic map "jump by k": from state i, the chain moves to $i+k$ when $1 \leq i+k \leq n$, and stays at i otherwise. We consider the stochastic matrices

$$A_n(p) = \sum_{k=-2}^{2} p_k\, Q_k^{(n)}.$$

The interior rows carry the pentadiagonal Toeplitz pattern $(\ldots, p_{-2}, p_{-1}, p_0, p_1, p_2, \ldots)$, while near the boundary the probability of each forbidden jump is transferred to the diagonal, exactly as in the tridiagonal case; this is the usual rejected-move convention for random walks in a confined domain (see, for example, [36]). We call these matrices *pentadiagonal Toeplitz-type stochastic matrices* and denote their class by $\mathcal{B}^{(2)}$. Setting $p_{-2} = p_2 = 0$ recovers the nearest-neighbor model of Section 5, so that $\mathcal{B}^{(1)} \subseteq \mathcal{B}^{(2)}$.

For fixed n, the matrices of order n in $\mathcal{B}^{(2)}$ form the convex hull of the five deterministic transition matrices $Q_{-2}^{(n)}, \ldots, Q_2^{(n)}$, so that $\Theta_n(\mathcal{B}^{(2)})$ is a hull spectrum in the sense of [18], although the generating set is no longer a group. Two features of the class should be kept in mind. First, $\mathcal{B}^{(2)}$ is not closed under the operation $S \mapsto S \oplus (1)$ of Section 2, and we do not know whether its eigenvalue regions are nested; accordingly, the realization function $d_{\mathcal{B}^{(2)}}$ records the first dimension in which an eigenvalue occurs, in the sense of Remark 2.2. Second, since $Q_0^{(n)} = I_n$, the generating set contains the identity, and the star-shapedness mechanism of Section 4 is available in the following exact form (compare [18, Lemma 2.6]). For $\mu \in \mathbb{C}$, we extend the notation of Section 4 by letting $[1,\ \mu] := \{(1-t)+t\mu : t \in [0,1]\}$ denote the segment joining 1 to μ.

Lemma 6.1. *Fix $n \geq 1$ and let Θ_n^0 denote the set of eigenvalues of the matrices $A_n(p)$ with $p_0 = 0$. Then*

$$\Theta_n(\mathcal{B}^{(2)}) = \bigcup_{\mu \in \Theta_n^0} [1, \mu].$$

Consequently, $\Theta_n(\mathcal{B}^{(2)})$ is star-shaped with respect to 1,

$$\max_{\lambda \in \Theta_n(\mathcal{B}^{(2)})} |\operatorname{Im}\lambda| = \max_{\mu \in \Theta_n^0} |\operatorname{Im}\mu|, \qquad \text{and} \qquad \Theta_n(\mathcal{B}^{(2)}) \cap \mathbb{R} = \Big[\min\Big(\Theta_n^0 \cap \mathbb{R}\Big),\ 1\Big].$$

The same statements hold, mutatis mutandis, when jumps of any bounded length are permitted.

Proof. If $p_0 = 1$, then $A_n(p) = I_n$ and $\sigma(A_n(p)) = \{1\}$. If $p_0 < 1$, put $s := 1 - p_0$ and $q_k := p_k/s$ for $k \neq 0$, $q_0 := 0$; then $A_n(p) = p_0 I_n + s\, A_n(q)$, whence

$$\sigma\big(A_n(p)\big) = p_0 + (1 - p_0)\, \sigma\big(A_n(q)\big).$$

As p_0 runs over $[0, 1]$ and μ over Θ_n^0, the value $p_0 + (1 - p_0)\mu$ runs over $\bigcup_{\mu \in \Theta_n^0} [1, \mu]$, which proves the displayed identity. Star-shapedness follows, since each segment issues from 1. For the second assertion, $\big|\operatorname{Im}\,(p_0 + (1 - p_0)\mu)\big| = (1 - p_0)|\operatorname{Im}\,\mu| \leq |\operatorname{Im}\,\mu|$, and the maximum over Θ_n^0 exists because Θ_n^0, being the set of eigenvalues of a compact family of matrices, is compact. For the third, a segment $[1,\ \mu]$ with μ nonreal meets the real axis only at 1, so the real points of the star are exactly the segments $[1,\ \mu]$ with $\mu \in \Theta_n^0 \cap \mathbb{R}$, whose union is the stated interval (which contains 1, since $1 \in \Theta_n^0$). Nothing in the argument uses the bound $|k| \leq 2$ on the jump lengths. □

In dimension 1, $\Theta_1(\mathcal{B}^{(2)}) = \{1\}$. In dimension 2, all jumps of length two are forbidden, so

$$A_2(p) = \begin{pmatrix} 1 - p_1 & p_1 \\ p_{-1} & 1 - p_{-1} \end{pmatrix}, \qquad \sigma\big(A_2(p)\big) = \{1,\ 1 - p_1 - p_{-1}\},$$

whence $\Theta_2(\mathcal{B}^{(2)}) = [0, 1] = \Theta_2(\mathcal{B}^{(1)})$. In particular, the spectra in dimensions 1 and 2 are real.

6.2 Dimension three

Lemma 6.1 allows us to assume $p_0 = 0$. For convenience, write

$$a := p_{-2}, \qquad b := p_{-1}, \qquad \mathrm{d} := p_1, \qquad e := p_2, \qquad a + b + d + e = 1,$$

so that

$$A_3 = \begin{pmatrix} a + b & d & e \\ b & a + e & d \\ a & b & d + e \end{pmatrix}.$$

Using $a + b + d + e = 1$, we find that $\det A_3 = ad + be - bd$ and

$$\det(tI - A_3) = (t - 1)\big(t^2 - ut + c\big), \qquad u := a + e, \quad c := ad + be - bd.$$

In particular, the nonreal eigenvalues of A_3, when they exist, are

$$\frac{u}{2} \pm i\sqrt{c - \frac{u^2}{4}}.$$

Their real part $u/2$ is thus nonnegative, and the largest possible imaginary part is found by maximizing $c - u^2/4$.

Theorem 6.2. *One has*

$$\Theta_3(\mathcal{B}^{(2)}) \cap \mathbb{R} = \left[-\frac{1}{2},\ 1\right] \qquad \text{and} \qquad \max_{\lambda \in \Theta_3(\mathcal{B}^{(2)})} |\operatorname{Im}\,\lambda| = \frac{1}{\sqrt{5}}.$$

The maximum is attained precisely by the two chains with

$$(p_{-2},\, p_1) = \left(\frac{2}{5},\, \frac{3}{5}\right) \qquad \text{or} \qquad (p_{-1},\, p_2) = \left(\frac{3}{5},\, \frac{2}{5}\right)$$

and all other $p_k = 0$, whose nonreal eigenvalues are $\frac{1 \pm i\sqrt{5}}{5}$.

Proof. We may restrict attention to the matrices A_3, by Lemma 6.1.

We begin with the imaginary parts. If A_3 has nonreal eigenvalues, their squared imaginary part is $y^2 = c - u^2/4$, and, since $-bd \le 0$,

$$y^2 \le ad + be - \frac{u^2}{4}.$$

Keeping a and e fixed, we have $b + d = 1 - u$. The linear expression $ad + be$ is largest when all of this remaining mass is assigned to the variable with the larger coefficient. Thus

$$ad + be \le (1-u)\max\{a, e\} \le (1-u)\,u,$$

and therefore

$$y^2 \le u(1-u) - \frac{u^2}{4} = u - \frac{5}{4}u^2 \le \frac{1}{5},$$

the last maximum being attained at $u = \frac{2}{5}$. To have equality, we must first have $u = \frac{2}{5}$ and $\max\{a, e\} = u$. Thus one of a, e is zero. If $a = u$ and $e = 0$, equality in the estimate for $ad + be$ requires $d = 1 - u$ and $b = 0$. The other possibility is obtained by reflection. Both choices give $c = \frac{6}{25}$ and eigenvalues $\frac{1}{5} \pm \frac{i}{\sqrt{5}}$. No other choices yield equality, and Lemma 6.1 shows that introducing a positive holding probability p_0 would strictly decrease the imaginary part. These are therefore the only maximizing chains.

We next determine the real eigenvalues. Since $\mathcal{B}^{(1)} \subseteq \mathcal{B}^{(2)}$, Theorem 5.3 already gives $[-\frac{1}{2}, 1] \subseteq \Theta_3(\mathcal{B}^{(2)})$. For the reverse inclusion, it is enough to consider A_3. If its two remaining eigenvalues are real, they are $\left(u \pm \sqrt{u^2 - 4c}\right)/2$. The smaller one is at least $-\frac{1}{2}$ precisely when $c \ge -\frac{2u+1}{4}$. But

$$c = ad + be - bd \ge -bd \ge -\frac{(b+d)^2}{4} = -\frac{(1-u)^2}{4}$$

and $(1-u)^2 \le 2u + 1$ for $0 \le u \le 1$. This proves the required lower bound. Equality is possible only when $u = 0$ and $b = d = \frac{1}{2}$, corresponding to the symmetric nearest-neighbor walk. That walk has $-\frac{1}{2}$ as an eigenvalue, so the endpoint is attained. □

6.3 Dimension four

We again take $p_0 = 0$, as permitted by Lemma 6.1, and retain the notation a, b, d, e. Set

$$u := a + e, \qquad v := b + d = 1 - u, \qquad \alpha := a - e, \qquad \beta := b - d,$$

so that $|\alpha| \le u$ and $|\beta| \le v$. With these notations,

$$A_4 = \begin{pmatrix} a+b & d & e & 0 \\ b & a & d & e \\ a & b & e & d \\ 0 & a & b & d+e \end{pmatrix}.$$

The characteristic polynomial factors as $\det(tI - A_4) = (t-1)Q(t)$, where

$$Q(t) = t^3 - ut^2 + \left[\frac{\beta^2 - \alpha\beta}{2} - u^2 + \frac{3}{2}u - \frac{1}{2}\right]t + \frac{(1-u)\alpha\beta - u\beta^2}{2}. \tag{6.1}$$

Let $x + iy$, $y \ne 0$, be a nonreal root of Q. Its conjugate is another root and, by Vieta's formulas, the third root is $u - 2x$. Writing

$$q := \beta^2, \qquad m := \alpha\beta, \qquad R := x^2 + y^2, \qquad B := (2x - u)\,R,$$

the coefficient of t and the constant term of (6.1) translate into

$$q - m = 2\left(y^2 - 3x^2 + 2ux + u^2 - \frac{3}{2}u + \frac{1}{2}\right) \qquad \text{and} \qquad (1-u)\,m - u\,q = 2B. \tag{6.2}$$

Since $|\beta| \le v$ and $|\alpha| \le u$, the quantities q and m satisfy the elementary bounds

$$0 \le \sqrt{q} \le v \qquad \text{and} \qquad |m| \le u\sqrt{q}.$$

When $u = \frac{1}{2}$, the cubic simplifies considerably:

$$Q(t) = \left(t - \frac{1}{2}\right)\left(t^2 + \frac{\beta^2 - \alpha\beta}{2}\right),$$

so the nonreal eigenvalues arising with $u = \frac{1}{2}$ are purely imaginary, that is, $x = 0$, and they satisfy

$$y^2 = \frac{\beta^2 - \alpha\beta}{2} \le \frac{|\beta|^2 + |\alpha||\beta|}{2} \le \frac{1}{4}.$$

The following estimates will be useful both for the real eigenvalues and for the imaginary parts.

Lemma 6.3. *Let $x + iy$, $y \neq 0$, be a nonreal eigenvalue of A_4, and let u, R, B be as above. Then*

$$y^2 \le 3x^2 - 2ux - u^2 + u \tag{6.3}$$

and

$$-u(1-u)^2 \le B \le \frac{u(1-u)^2}{8}. \tag{6.4}$$

Proof. Since $m \ge -u\sqrt{q}$ and $\sqrt{q} \le v$,

$$q - m \leqq +u\sqrt{q} \le v^2 + uv = v = 1 - u,$$

and inserting this into the first relation of (6.2) gives (6.3). For the second inequality, recall that $2B = vm - uq$. For the lower bound, $m \ge -u\sqrt{q}$ and $q \le v^2$ give

$$2B \ge -uv\sqrt{q} - uq \ge -uv^2 - uv^2 = -2uv^2.$$

For the upper bound, $m \le u\sqrt{q}$ gives

$$2B \le u\sqrt{q}\left(v - \sqrt{q}\right) \le \frac{uv^2}{4},$$

since $w(v - w) \le v^2/4$ for $0 \le w \le v$. □

Theorem 6.4

$$\Theta_4(\mathcal{B}^{(2)}) \cap \mathbb{R} = \left[-\frac{1}{\sqrt{2}}, 1\right].$$

In particular, allowing jumps of length two does not enlarge the real slice of the spectral region in dimension four.

Proof. Theorem 5.3 gives the inclusion $[-1/\sqrt{2}, 1] \subseteq \Theta_4(\mathcal{B}^{(2)})$, since $\mathcal{B}^{(1)} \subseteq \mathcal{B}^{(2)}$. For the other inclusion, Lemma 6.1 reduces the problem to Q. We shall show that Q has no root smaller than $-1/\sqrt{2}$. We will also determine when that endpoint is attained. Rewrite (6.1) as

$$Q(t) = t^3 - ut^2 + \left(-u^2 + \frac{3}{2}u - \frac{1}{2}\right)t + \frac{1 - u - t}{2}\,\alpha\beta + \frac{t - u}{2}\,\beta^2.$$

Fix $h \ge \frac{1}{\sqrt{2}}$ and evaluate at $t = -h$. The coefficient $\frac{1-u+h}{2}$ of $\alpha\beta$ is positive and $\alpha\beta \le u|\beta| \le u(1-u)$, while the coefficient $\frac{-h-u}{2}$ of β^2 is negative, so that term may be discarded. A short computation then gives

$$Q(-h) - \left(-h^3 + \frac{h}{2}\right) \le \frac{u}{2}\left[-2h^2 + hu - 2h + (1-u)^2\right] \le \frac{u}{2}\left(-2h^2 - h + 1\right) = -\frac{u}{2}\,(2h-1)(h+1) \le 0,$$

using $hu \le h$ and $(1-u)^2 \le 1$. Hence

$$Q(-h) \le -h\Big(h^2 - \frac{1}{2}\Big) \le 0,$$

with strict inequality whenever $h > \frac{1}{\sqrt{2}}$. Thus Q has no root below $-1/\sqrt{2}$.

At $h = 1/\sqrt{2}$, the factor $(2h-1)(h+1)$ is strictly positive, so equality in the preceding bound forces $u = 0$. We then have $a = e = \alpha = 0$, and

$$Q(-h) = -\frac{h}{2}\,\beta^2.$$

Consequently, $-1/\sqrt{2}$ is a root only when $\beta = 0$, or equivalently $b = d = 1/2$. This is the symmetric nearest-neighbor walk, which has the required eigenvalue by Lemma 5.2. □

Corollary 6.5. *For every real* $\lambda \in \Big[-\frac{1}{\sqrt{2}}, 1\Big]$,

$$d_{\mathcal{B}^{(2)}}(\lambda) = d_{\mathcal{B}^{(1)}}(\lambda) = \left\lceil \frac{\pi}{\arccos(-\lambda)} \right\rceil.$$

Proof. Fix $\lambda \in [-1/\sqrt{2}, 1]$ and set $k := d_{\mathcal{B}^{(1)}}(\lambda)$, so that $k \le 4$ by Corollary 5.5, and $d_{\mathcal{B}^{(2)}}(\lambda) \le k$ because $\mathcal{B}^{(1)} \subseteq \mathcal{B}^{(2)}$. If λ had a pentadiagonal realization in some dimension $n < k$, then $n \le 3$; but in dimensions 1, 2 and 3 the real slice of the pentadiagonal region agrees with the tridiagonal region, by the computations at the end of Subsection 6.1 and by Theorem 6.2, so that λ would also have a tridiagonal realization in dimension n, contrary to the definition of k. Hence $d_{\mathcal{B}^{(2)}}(\lambda) = k$. □

Proposition 6.6. *Every* $\lambda \in \Theta_4(\mathcal{B}^{(2)})$ *satisfies*

$$|\operatorname{Im}\lambda| \le \frac{1}{\sqrt{3}}.$$

Proof. By Lemma 6.1 we may take $p_0 = 0$; let $x + iy$, $y > 0$, be a nonreal eigenvalue of A_4, with u, $v = 1-u$, R, B as above. We consider four cases. The first three give the stronger estimate $y^2 \le \frac{1}{4}$; only the last requires the bound $y^2 \le \frac{1}{3}$.

Case $u = \frac{1}{2}$. As noted before Lemma 6.3, then $x = 0$ and $y^2 \le \frac{1}{4}$.

Case $B \ge 0$, $u \ne \frac{1}{2}$. Then $2x \ge u$, so $x \ge u/2$. Suppose, for a contradiction, that $y^2 > \frac{1}{4}$. Then $R > \frac{1}{4}$, and the upper bound in (6.4) yields

$$2x - u = \frac{B}{R} \le \frac{u(1-u)^2}{8R} \le \frac{u(1-u)^2}{2}, \qquad \text{whence} \qquad x \le \bar{x}(u) := \frac{u}{2} + \frac{u(1-u)^2}{4}.$$

The right-hand side of (6.3) is increasing in x for $x \ge u/3$, so

$$y^2 \le 3\,\bar{x}(u)^2 - 2u\,\bar{x}(u) + u(1-u) =: \psi(u).$$

We claim that $\psi(u) \le \frac{1}{4}$ for every $u \in [0,1]$, which produces the desired contradiction. Indeed, $\bar{x}(u) = \frac{u}{4}(2+v^2)$ and $3\bar{x}(u) - 2u = \frac{u}{4}(3v^2 - 2)$, whence the identity

$$\psi(u) = \bar{x}(u)\big(3\bar{x}(u) - 2u\big) + uv = \frac{u^2(2+v^2)(3v^2-2)}{16} + uv.$$

If $3v^2 \le 2$, the first term is nonpositive, so $\psi(u) \le uv \le \frac{1}{4}$. If $3v^2 > 2$, then $v > \sqrt{2/3} > \frac{4}{5}$, so $u < \frac{1}{5}$; since moreover $2 + v^2 \le 3$ and $3v^2 - 2 \le 1$,

$$\psi(u) \le \frac{3u^2}{16} + uv < \frac{3}{400} + \frac{1}{5} = \frac{83}{400} < \frac{1}{4}.$$

Hence $y^2 \le \frac{1}{4}$ in this case.

Case $B < 0$ and $r := u - 2x \in (0, u]$. Substituting $x = (u - r)/2$ into (6.3) gives

$$y^2 \le F_u(r) := \frac{3r^2 - 2ur - 5u^2 + 4u}{4}. \tag{6.5}$$

The function F_u is convex, and

$$F_u(0) = \frac{4u - 5u^2}{4} \le \frac{1}{5}, \qquad F_u(u) = u(1-u) \le \frac{1}{4},$$

so $F_u \le \frac{1}{4}$ on $[0, u]$.

Case $B < 0$ and $r > u$. The lower bound in (6.4) reads $rR \le u(1-u)^2$. Suppose $y^2 > \frac{1}{3}$. Then $r\,y^2 \le rR \le u(1-u)^2$ gives $r < 3u(1-u)^2$, and, since also $r > u$, (6.5) yields

$$4y^2 \le 3r^2 - 2ur - 5u^2 + 4u < 27u^2(1-u)^4 - 2u^2 - 5u^2 + 4u = 27u^2(1-u)^4 - 7u^2 + 4u.$$

Since $u^2(1-u)^4 \le \frac{16}{729}$ (maximum at $u = \frac{1}{3}$) and $-7u^2 + 4u \le \frac{4}{7}$ (maximum at $u = \frac{2}{7}$),

$$4y^2 < \frac{16}{27} + \frac{4}{7} = \frac{220}{189} < \frac{4}{3},$$

contradicting $y^2 > \frac{1}{3}$. Hence $y^2 \le \frac{1}{3}$, completing the proof. □

We now look for chains attaining the largest possible imaginary part. Consider first the *two-jump chains*

$$p_{-2} = u, \qquad p_1 = 1 - u, \qquad (0 \le u \le 1),$$

together with their reflections $p_2 = u$, $p_{-1} = 1 - u$. For these chains, $(\alpha, \beta) = \big(u, -(1-u)\big)$, and (6.1) becomes

$$Q(t) = t^3 - ut^2 + u(1-u)\,t - u(1-u)^2. \tag{6.6}$$

The cubic (6.6) becomes tractable after a change of parameter. Set

$$u = u(s) := \frac{s^3}{\delta(s)}, \qquad \delta(s) := s^3 + s^2 - s + 1, \qquad (s > 0),$$

where the denominator is positive for $s > 0$, since $\delta(s) = s^3 + (s - \frac{1}{2})^2 + \frac{3}{4}$. Substituting into (6.6) and clearing denominators, one finds that

$$Q(t) = \left(t - \frac{s(s^2 - s + 1)}{\delta(s)}\right)\left(t^2 - \frac{s(s-1)}{\delta(s)}\,t + \frac{s^2(s^2 - s + 1)}{\delta(s)^2}\right).$$

The discriminant of the quadratic factor is $-s^2(3s^2 - 2s + 3)/\delta(s)^2$, which is negative for all $s > 0$, so that its two roots are the nonreal conjugate pair

$$\lambda_{\pm}(s) = \frac{s(s-1) \pm is\sqrt{3s^2 - 2s + 3}}{2\,\delta(s)}. \tag{6.7}$$

Theorem 6.7. *The quintic*

$$P(s) = 3s^5 - 3s^4 + 8s^3 - 4s^2 + 3s - 3$$

has a unique positive root s_*, and $s_* \in (0, 1)$. With $\delta(s) = s^3 + s^2 - s + 1$ and $\lambda_{\pm}(s)$ as in (6.7), set

$$u_* := u(s_*) = \frac{s_*^3}{\delta(s_*)}, \qquad M := \operatorname{Im} \lambda_+(s_*) = \frac{s_*\sqrt{3s_*^2 - 2s_* + 3}}{2\,\delta(s_*)}.$$

Then $u_* \in \left(0, \frac{1}{2}\right)$,

$$\max_{\lambda \in \Theta_4(\mathcal{B}^{(2)})} |\operatorname{Im} \lambda| = M, \qquad \text{and} \qquad \frac{1}{2} < M \le \frac{1}{\sqrt{3}}.$$

The maximum is attained precisely by the two chains $p_{-2} = u_*$, $p_1 = 1 - u_*$ *and* $p_2 = u_*$, $p_{-1} = 1 - u_*$ *(all other* p_k *being equal to 0), whose nonreal eigenvalues are* $\lambda_\pm(s_*)$.

Proof. By Lemma 6.1, we may first take $p_0 = 0$. Let $x + iy$, $y > 0$, be a nonreal eigenvalue of A_4, with u, $v = 1 - u$, R and B as before, and put $r := u - 2x$. The proof of Proposition 6.6 gives $y^2 \le \frac{1}{4}$ except possibly when $B < 0$ and $r > u$, so we concentrate on that case. There, the lower bound in (6.4) reads $rR \le u(1-u)^2$; this forces $0 < u < 1$ and, since $R = x^2 + y^2$ with $x = (u - r)/2$, it provides a second upper bound for y^2 alongside (6.5), namely

$$y^2 \le G_u(r) := \frac{u(1-u)^2}{r} - \frac{(u-r)^2}{4}. \tag{6.8}$$

For fixed u, the function F_u is strictly increasing on (u, ∞) while G_u is strictly decreasing there, since

$$F_u'(r) = \frac{3r - u}{2} > 0 \qquad \text{and} \qquad G_u'(r) = -\frac{u(1-u)^2}{r^2} - \frac{r-u}{2} < 0.$$

If $u \ge \frac{1}{2}$, then $y^2 \le G_u(r) < G_u(u) = (1-u)^2 \le \frac{1}{4}$, and nothing new happens. If $0 < u < \frac{1}{2}$, then $F_u(u) = u(1-u) < (1-u)^2 = G_u(u)$, and since $F_u(r) \to \infty$ and $G_u(r) \to -\infty$ as $r \to \infty$, the two graphs cross at a unique point $r_c(u) > u$, which is where $\min\{F_u, G_u\}$ is largest. Hence

$$y^2 \le \min\{F_u(r), G_u(r)\} \le F_u\big(r_c(u)\big) = G_u\big(r_c(u)\big) =: C(u), \qquad \Big(0 < u < \frac{1}{2}\Big).$$

The two-jump chains attain the bound $C(u)$. Indeed, the map $s \mapsto u(s) = s^3/\delta(s)$ satisfies $u'(s) = s^2(s^2 - 2s + 3)/\delta(s)^2 > 0$ for $s > 0$, tends to 0 as $s \downarrow 0$ and takes the value $\frac{1}{2}$ at $s = 1$, so it maps $(0, 1)$ bijectively onto $(0, \frac{1}{2})$. For the two-jump chain with parameter $u = u(s)$, $0 < s < 1$, we have $q = (1-u)^2$ and $m = -u(1-u)$, so that equality holds in (6.3) and in the lower bound of (6.4), hence in both (6.5) and (6.8); moreover (6.7) gives $x < 0$, that is, $r > u$. The eigenvalue $\lambda_+(s)$ therefore sits exactly at $r = r_c(u)$, and

$$C\big(u(s)\big) = H(s), \qquad H(s) := \big(\operatorname{Im} \lambda_+(s)\big)^2 = \frac{s^2(3s^2 - 2s + 3)}{4\,\delta(s)^2}.$$

Since every other case yields $y^2 \le \frac{1}{4}$, the maximum of y^2 over the class with $p_0 = 0$ is the larger of $\frac{1}{4}$ and $\sup_{0<s<1} H(s)$.

A direct computation gives $H'(s) = -sP(s)/(2\delta(s)^3)$, where P is the quintic of the statement. The polynomial P is strictly increasing on $(0, \infty)$, because $P'(s) = 3s^2(5s^2 - 4s + 1) + (21s^2 - 8s + 3)$ is a sum of two quadratics with positive leading coefficients and negative discriminants; as $P(0) = -3$ and $P(1) = 4$, its unique positive root s_* lies in $(0, 1)$, and consequently $u_* = u(s_*) \in (0, \frac{1}{2})$. Thus H increases on $(0, s_*]$ and decreases on $[s_*, 1)$, and its maximum $H(s_*) = M^2$ exceeds $\frac{1}{4}$, since already $H(3/4) = \frac{1{,}836}{6{,}241} > \frac{1}{4}$. The largest imaginary part over the class with $p_0 = 0$ is therefore M, attained by the two-jump chain with $u = u_*$; the inequality $M \le 1/\sqrt{3}$ is Proposition 6.6.

It remains to identify all chains attaining the maximum. If $y = M$, then $y^2 > \frac{1}{4}$, so $0 < u < \frac{1}{2}$ and $r > u$, and the inequalities $y^2 \le \min\{F_u(r), G_u(r)\} \le C(u) \le M^2$ are all equalities. The uniqueness of the maximum of H gives $u = u_*$, the strict monotonicity of F_u and G_u gives $r = r_c(u)$, and in particular equality holds in (6.5), hence throughout the chain of inequalities $q - m \le q + u\sqrt{q} \le v^2 + uv = v$ of the proof of Lemma 6.3. As $w \mapsto w^2 + uw$ is strictly increasing on $[0, \infty)$ and $0 \le \sqrt{q} \le v$, the second equality forces $\sqrt{q} = v$, and the first then gives $m = -uv$; recalling that $q = \beta^2$ and $m = \alpha\beta$, we obtain $(\alpha, \beta) = (u, -v)$ or $(\alpha, \beta) = (-u, v)$, which are precisely the two reflected chains of the statement. Finally, a chain with $p_0 > 0$ cannot be extremal, since Lemma 6.1 gives $|\operatorname{Im} \lambda| \le (1 - p_0)M < M$ for its eigenvalues. The extremal eigenvalues are $\lambda_\pm(s_*)$ by (6.7). □

Approximate values of s_*, u_*, M and $\lambda_\pm(s_*)$ are recorded in Remark 6.10 below.

6.4 An eigenvalue with minimal dimension of realization five

The preceding bounds allow us to determine the minimal realization dimension of an explicit nonreal eigenvalue.

Example 6.8. For $n = 5$, take

$$p_{-1} = \frac{1}{2}, \qquad p_2 = \frac{1}{2},$$

with all other $p_k = 0$. The resulting pentadiagonal Toeplitz-type stochastic matrix is

$$A = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

and a direct computation gives

$$\det(tI - A) = \frac{1}{16}(t-1)(2t-1)\big(8t^3 + 2t - 1\big).$$

The cubic factor $8t^3 + 2t - 1$ has derivative $24t^2 + 2 > 0$; hence it has exactly one real root, and its two remaining roots form a nonreal complex conjugate pair.

Theorem 6.9. *Let $z = x_* + iy_*$ (with $y_* > 0$) and $\overline{z}$ be the nonreal roots of $8t^3 + 2t - 1$; explicitly,*

$$x_* = -\frac{w_0}{4}, \qquad y_* = \frac{1}{2}\sqrt{1 + \frac{3w_0^2}{4}}, \qquad \text{where} \qquad w_0 = \sqrt[3]{\sqrt{\frac{31}{108}} + \frac{1}{2}} - \sqrt[3]{\sqrt{\frac{31}{108}} - \frac{1}{2}}$$

is the real root of $w^3 + w - 1$. Then

$$d_{\mathcal{B}^{(2)}}(z) = d_{\mathcal{B}^{(2)}}(\overline{z}) = 5.$$

Proof. The matrix of Example 6.8 realizes z and $\overline{z}$ in dimension 5, so $d_{\mathcal{B}^{(2)}}(z) \le 5$; it remains to show that $z \notin \Theta_n(\mathcal{B}^{(2)})$ for $n \le 4$, and by conjugation symmetry it suffices to treat z.

We first justify the explicit expressions. The substitution $t = w/2$ turns $8t^3 + 2t - 1$ into $w^3 + w - 1$, which is strictly increasing on $\mathbb{R}$ and whose unique real root is given by Cardano's formula as the number w_0 of the statement. Since $w_0^3 + w_0 = 1$, we have $w^3 + w - 1 = (w - w_0)\big(w^2 + w_0 w + w_0^2 + 1\big)$, so the nonreal roots of $w^3 + w - 1$ are $-\frac{w_0}{2} \pm i\sqrt{1 + \frac{3w_0^2}{4}}$; halving them gives the stated values of x_* and y_*.

We next bound y_* from below. Since $w^3 + w - 1$ is increasing and takes the value $-\frac{1}{27} < 0$ at $w = \frac{2}{3}$, we have $w_0 > \frac{2}{3}$, and therefore

$$y_*^2 = \frac{1}{4} + \frac{3w_0^2}{16} > \frac{1}{4} + \frac{1}{12} = \frac{1}{3}, \qquad \text{that is,} \qquad y_* > \frac{1}{\sqrt{3}}.$$

Now compare with the regions of small dimension. The spectra in dimensions 1 and 2 are real. In dimension 3, Theorem 6.2 gives $|\operatorname{Im}\lambda| \le \frac{1}{\sqrt{5}} < \frac{1}{\sqrt{3}}$ for every $\lambda \in \Theta_3(\mathcal{B}^{(2)})$. In dimension 4, Proposition 6.6 gives $|\operatorname{Im}\lambda| \le \frac{1}{\sqrt{3}}$, and the inequality $y_* > \frac{1}{\sqrt{3}}$ is strict. Hence z belongs to none of the regions $\Theta_n(\mathcal{B}^{(2)})$ with $n \le 4$, and $d_{\mathcal{B}^{(2)}}(z) = 5$. □

Remark 6.10. The quantities appearing in Theorem 6.7 are defined exactly, and their decimal expansions play no role in the proofs; for the reader's orientation, their values rounded to seven places are

$$s_* \approx 0.7361724, \qquad u_* \approx 0.3311640, \qquad M \approx 0.5425638, \qquad \lambda_{\pm}(s_*) \approx -0.0806073 \pm 0.5425638\, i.$$

These values can be certified by rational brackets: since P is strictly increasing on $(0,\infty)$ and $P(0.7361) < 0 < P(0.7362)$, we have $0.7361 < s_* < 0.7362$, and narrower brackets, inserted into the explicit formulas for u_*, M and $\lambda_{\pm}$, bound the remaining quantities to any desired accuracy. Likewise, for Theorem 6.9, one finds $\omega_0 \approx 0.6823278$ and

$$z \approx -0.1705820 + 0.5807707\, i.$$

It is worth noting how close the eigenvalue of Theorem 6.9 comes to the bound used to exclude dimension four: its imaginary part is approximately 0.5808, against $1/\sqrt{3} \approx 0.5774$ in Proposition 6.6, whereas the exact maximum $M \approx 0.5426$ of Theorem 6.7 leaves a more comfortable margin.

6.5 Two conjectures

The results of this section suggest two conjectures; an exact description of the full boundary of $\Theta_4(\mathcal{B}^{(2)})$, in particular, remains out of reach of the present methods.

Conjecture 6.11. For every $n \geq 1$,

$$\Theta_n(\mathcal{B}^{(2)}) \cap \mathbb{R} = \Big[-\cos\Big(\frac{\pi}{n}\Big), 1\Big].$$

The conjecture holds for $n \leq 4$ by Theorems 6.2 and 6.4. If it holds in every dimension, then the realization functions of $\mathcal{B}^{(2)}$ and $\mathcal{B}^{(1)}$ agree on the real axis, so that allowing jumps of length two, which certainly changes the regions off the axis, would never reduce the dimension needed to realize a real eigenvalue.

Conjecture 6.12. For every $n \geq 3$, the maximum of $|\mathrm{Im}\ \lambda|$ over $\Theta_n(\mathcal{B}^{(2)})$ is attained by a two-jump chain $p_{-2} = u$, $p_1 = 1 - u$, up to reflection.

Theorems 6.2 and 6.7 establish Conjecture 6.12 in dimensions three and four; it is open for $n \geq 5$. A proof in dimension five would reduce the search for the largest imaginary part in that dimension to the one-parameter family of two-jump chains, and would be the natural first step towards an analogue of Theorem 6.9 one dimension higher, since such an analogue requires rigorous upper bounds on the imaginary parts in every dimension up to five, together with an explicit six-dimensional chain possessing an eigenvalue that exceeds all of them.

More generally, one may let $\mathcal{B}^{(m)}$ denote the class obtained by allowing jumps of length at most m under the same boundary convention. Lemma 6.1 applies verbatim to each of these classes, but we do not know how their eigenvalue regions depend on m, nor whether the maximizers of the imaginary part can again be found among chains that use only two jump lengths.

7 Conclusions and further research questions

The main question left open by this work is the minimal dimension of realization problem for the full class Σ of stochastic matrices. Although the Karpelevič regions Θ_n are known, the smallest dimension in which a prescribed eigenvalue occurs does not seem to admit a closed description at present, and we conclude by indicating one way of organizing the question.

Since every Θ_n with $n \geq 2$ is closed and star-shaped with respect to the origin, it is determined by its radial boundary function

$$R_n(\theta) := \max\{\rho \geq 0 :\ \rho e^{i\theta} \in \Theta_n\}, \qquad (\theta \in \mathbb{R}),$$

and since the regions are nested, the realization function of Σ is recovered from these functions: $d_\Sigma(1) = 1$ and, for $\lambda = re^{i\theta} \neq 1$ with $r \geq 0$,

$$d_\Sigma(re^{i\theta}) = \min\{n \geq 2 : r \leq R_n(\theta)\},$$

with the usual convention that the minimum of the empty set is ∞. The radial boundary functions are thus the natural objects to study. Kirkland, Laffey and Šmigoc [10] parametrized the boundary of the Karpelevič regions by means of Farey pairs, and their results provide a starting point: it would be of interest to obtain explicit expressions for $R_n(\theta)$ along particular rays, to investigate the asymptotic behavior of $R_n(\theta)$ as $n \to \infty$, and to determine whether this approach leads to a practical description of d_Σ.

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