

The Sharp Constant in the Mashregghi–Ransford Inequality

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A Question About Moments

- Let μ be a complex measure, and define

$$a_n := \int \zeta^n d\mu(\zeta), \quad b_n := \int (\zeta + 1)^n d\mu(\zeta), \quad c_n := \int (\zeta - 1)^n d\mu(\zeta).$$

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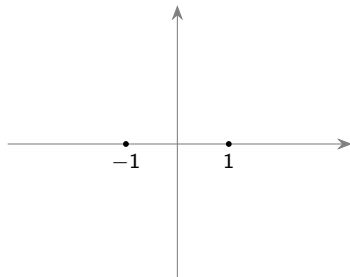
Question

If both b_n and c_n grow like $O(\beta^n)$ for some $\beta > 1$, how large can the moments a_n (relative to 0) be?

A Heuristic

If μ is supported on a compact set K , then

$$\left| \int (\zeta - c)^n d\mu(\zeta) \right| \leq \|\mu\| \max_{\zeta \in K} |\zeta - c|^n.$$

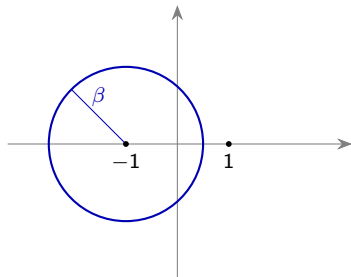


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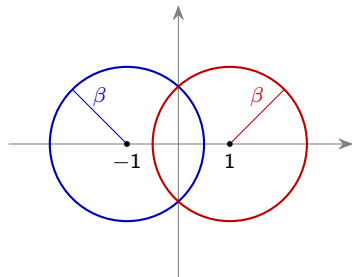


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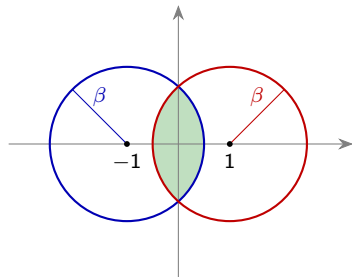


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- $\Rightarrow \text{supp } \mu \text{ lies in } L := \overline{D}(-1, \beta) \cap \overline{D}(1, \beta).$



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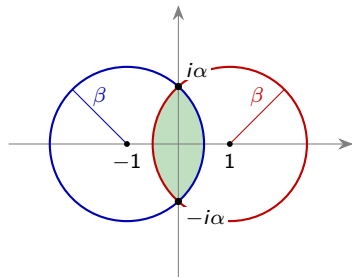
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$$\Rightarrow \text{supp } \mu \text{ lies in } L := \overline{D}(-1, \beta) \cap \overline{D}(1, \beta).$$

$$\Rightarrow |a_n| \leq \|\mu\| \max_{\zeta \in L} |\zeta|^n = \|\mu\| \alpha^n,$$

where $\alpha := \sqrt{\beta^2 - 1}$.



A Different Point of View

- By the binomial theorem,

$$b_n = \int (\zeta + 1)^n d\mu(\zeta) = \sum_{k=0}^n \binom{n}{k} \int \zeta^k d\mu(\zeta) = \sum_{k=0}^n \binom{n}{k} a_k,$$

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$\implies (b_n)$ and (c_n) are just two *binomial transforms* of (a_n) !

Remark

From now on, (a_n) is an arbitrary sequence of complex numbers, and no measure is needed.

The Mashreghi–Ransford Inequality

Theorem (Mashreghi–Ransford; 2005)

Let $\beta > 1$, let $\alpha := \sqrt{\beta^2 - 1}$, and let (a_n) be a sequence of complex numbers. If $b_n, c_n = O(\beta^n)$, then $a_n = O(\alpha^n)$. More precisely,

$$\limsup_{n \rightarrow \infty} \frac{|a_n|}{\alpha^n} \leq 2 \left(\limsup_{n \rightarrow \infty} \frac{|b_n|}{\beta^n} \right)^{1/2} \left(\limsup_{n \rightarrow \infty} \frac{|c_n|}{\beta^n} \right)^{1/2}.$$

Is the Constant Optimal?

Question

What is the smallest universal constant κ such that

$$\limsup_{n \rightarrow \infty} \frac{|a_n|}{\alpha^n} \leq \kappa \left(\limsup_{n \rightarrow \infty} \frac{|b_n|}{\beta^n} \right)^{1/2} \left(\limsup_{n \rightarrow \infty} \frac{|c_n|}{\beta^n} \right)^{1/2}$$

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- Mashreghi and Ransford showed that $\frac{2}{\sqrt{3}} \leq \kappa \leq 2$.

The Example

- Take $\beta = 2$, so that $\alpha = \sqrt{3}$, and $a_n := (i\sqrt{3})^n - (-i\sqrt{3})^n$.

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$$a_n = (\sqrt{3})^n (i^n - (-i)^n) = 2i (\sqrt{3})^n \sin(n\pi/2),$$

$$b_n = (1 + i\sqrt{3})^n - (1 - i\sqrt{3})^n = 2^{n+1} i \sin(n\pi/3),$$

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Hence

$$\limsup_{n \rightarrow \infty} \frac{|b_n|}{2^n} = \limsup_{n \rightarrow \infty} \frac{|c_n|}{2^n} = \sqrt{3} \quad \text{and} \quad \limsup_{n \rightarrow \infty} \frac{|a_n|}{(\sqrt{3})^n} = 2.$$

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$$\implies 2 \leq \kappa (\sqrt{3})^{1/2} (\sqrt{3})^{1/2} = \kappa \sqrt{3} \iff \kappa \geq \frac{2}{\sqrt{3}}.$$

Main Result

Theorem (B.; 2026+)

Let $\beta > 1$, let $\alpha := \sqrt{\beta^2 - 1}$, and let (a_n) be a sequence of complex numbers. If $b_n, c_n = O(\beta^n)$, then

$$\limsup_{n \rightarrow \infty} \frac{|a_n|}{\alpha^n} \leq \frac{2}{\sqrt{3}} \left(\limsup_{n \rightarrow \infty} \frac{|b_n|}{\beta^n} \right)^{1/2} \left(\limsup_{n \rightarrow \infty} \frac{|c_n|}{\beta^n} \right)^{1/2}.$$

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Moreover, the constant $2/\sqrt{3}$ is optimal.

Goal of the talk: show that the sharp constant is indeed $2/\sqrt{3}$.

Step 0: Real Sequences

- It suffices to prove the theorem for **real** sequences (a_n) .

Normalization

- Let $p := \alpha/\beta$ and $q := 1/\beta$. Then $p, q > 0$ and $p^2 + q^2 = 1$, so there is $0 < \phi < \pi/2$ such that

$$p = \sin \phi \quad \text{and} \quad q = \cos \phi.$$

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- Let

$$x_n := \frac{a_n}{\alpha^n}, \quad y_n := \frac{b_n}{\beta^n}, \quad z_n := (-1)^n \frac{c_n}{\beta^n},$$

and consider their *exponential generating functions*

$$F(w) := \sum_{n \geq 0} x_n \frac{w^n}{n!}, \quad G(w) := \sum_{n \geq 0} y_n \frac{w^n}{n!}, \quad H(w) := \sum_{n \geq 0} z_n \frac{w^n}{n!}.$$

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New goal

With $B := \limsup |y_n|$, $C := \limsup |z_n|$, show that $\limsup |x_n| \leq \frac{2}{\sqrt{3}} \sqrt{BC}$.

Key Identities

- $p = \sin \phi = \alpha/\beta$, $q = \cos \phi = 1/\beta$
- $x_n = a_n/\alpha^n$, $y_n = b_n/\beta^n$, $z_n = (-1)^n c_n/\beta^n$
- Binomial theorem: $G(w) = e^{qw}F(pw)$ and $H(w) = e^{qw}F(-pw)$

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- Real coefficients: $F(\overline{w}) = \overline{F(w)}$, and similarly for G and H .
- Growth: since (y_n) and (z_n) are bounded, G and H are entire and

$$|G(w)| \leq (B + o(1))e^{|w|}, \quad |H(w)| \leq (C + o(1))e^{|w|} \quad (w \in \mathbb{C}).$$

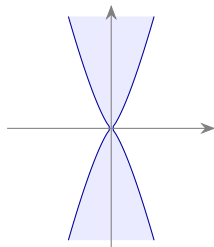
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- $B = \limsup |y_n|$, $C = \limsup |z_n|$

Theorem (Mashreghi–Ransford; 2005)

As $|z| \rightarrow \infty$,

$$|F(z)| \leq \begin{cases} (\sqrt{BC} + o(1)) e^{|\operatorname{Im} z|} & \text{if } |\operatorname{Re} z| < |z|^{3/4}, \\ e^{|z| - \frac{1}{2}|z|^{1/2} + O(1)} & \text{if } |\operatorname{Re} z| \geq |z|^{3/4}. \end{cases}$$

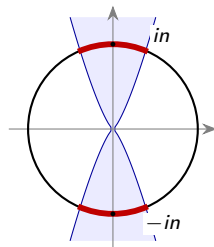


Cauchy's Formula and Two Points Estimate

Idea: By Cauchy's formula on the circle $|w| = n$,

$$x_n = \frac{n!}{2\pi n^n} \int_{-\pi}^{\pi} F(ne^{it}) e^{-int} dt,$$

and only the neighborhoods of the two points $\pm in$ contribute as $n \rightarrow \infty$.

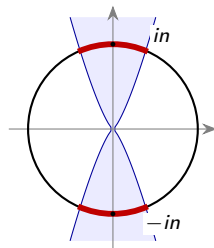


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Lemma (Two points estimate)

Let $\eta_n := \arcsin(n^{-1/4})$. If, for some $K \geq 0$,

$$\sup_{|\theta| \leq \eta_n} e^{-n \cos \theta} \left(|F(-ine^{i\theta})| + |F(ine^{-i\theta})| \right) \leq K + o(1),$$

then $\limsup_{n \rightarrow \infty} |x_n| \leq K$.

Where Does the Constant 2 Come From?

- For $|\theta| \leq \eta_n$, $z = \pm i n e^{\mp i \theta}$ has $|\operatorname{Re} z| \leq |z|^{3/4}$ and $|\operatorname{Im} z| = n \cos \theta$, so the contribution of each point is at most $(\sqrt{BC} + o(1))e^{n \cos \theta}$.

Where Does the Constant 2 Come From?

- For $|\theta| \leq \eta_n$, $z = \pm ine^{\mp i\theta}$ has $|\operatorname{Re} z| \leq |z|^{3/4}$ and $|\operatorname{Im} z| = n \cos \theta$, so the contribution of each point is at most $(\sqrt{BC} + o(1))e^{n \cos \theta}$.

Adding the two contributions,

$$|F(-ine^{i\theta})| + |F(ine^{-i\theta})| \leq (2\sqrt{BC} + o(1))e^{n \cos \theta}.$$

$\Rightarrow$ The lemma with $K = 2\sqrt{BC}$ gives the Mashreghi–Ransford inequality.

How to Improve it?

Theorem (Mashreghi–Ransford; 2005)

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What we will show

Each point in the neighborhood of $\pm in$ only contributes $\frac{1}{\sqrt{3}}\sqrt{BC}$, uniformly for $|\theta| \leq \eta_n$:

$$|F(\pm ine^{\mp i\theta})| \leq \left(\frac{1}{\sqrt{3}}\sqrt{BC} + o(1) \right) e^{n \cos \theta}.$$

An Important Observation

- $p = \sin \phi = \alpha/\beta$, $q = \cos \phi = 1/\beta$
- $x_n = a_n/\alpha^n$, $y_n = b_n/\beta^n$, $z_n = (-1)^n c_n/\beta^n$

Lemma (B.; 2026+)

Let $0 < \phi < \pi/2$ and $q = \cos \phi$. Suppose that E is entire and that, for some $M > 0$,

$$|E(w)| \leq Me^{|w|} \quad \text{and} \quad |e^{-2qw} E(w)| \leq Me^{|w|} \quad (w \in \mathbb{C}).$$

Then

$$E(te^{\pm 3i\phi}) = o(e^t) \quad (t \rightarrow \infty).$$

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Application

Since $e^{-2qw} G(w) = H(-w)$ and $e^{-2qw} H(w) = G(-w)$, this applies to $E = G$ and to $E = H$.

The Main New Ingredient

By the previous lemma, $G(te^{\pm 3i\phi}) = o(e^t)$ and $H(te^{\pm 3i\phi}) = o(e^t)$.

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Lemma (B.; 2026+)

Let $0 < \phi < \pi/2$, let $(s_n)_{n \geq 0}$ be a bounded **real** sequence, and let $E(w) := \sum_{n \geq 0} s_n \frac{w^n}{n!}$. If

$$e^{-t} E(te^{\pm 3i\phi}) \longrightarrow 0 \quad (t \rightarrow \infty),$$

then

$$\limsup_{t \rightarrow \infty} e^{-t} |E(te^{\pm i\phi})| \leq \frac{1}{\sqrt{3}} \limsup_{n \rightarrow \infty} |s_n|.$$

Proof of the Directional Estimate (1/3)

For $t > 0$, $\theta \in \mathbb{R}$ and $k \geq 1$, let

$$w_n := e^{-t} \frac{t^n}{n!} \quad \text{and} \quad h_{n,k} := \cos(k(n\phi + \theta)).$$

Then

$$\sum_{n \geq 0} w_n s_n h_{n,k} = e^{-t} \operatorname{Re} \left(e^{ik\theta} E(te^{ik\phi}) \right).$$

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For $k = 1$, we can choose θ (depending on t) such that

$$\left| \sum_{n \geq 0} w_n s_n h_{n,1} \right| = e^{-t} |E(te^{i\phi})|.$$

$\implies$ It suffices to bound the left-hand side *uniformly* in θ .

Proof of the Directional Estimate (2/3)

Kill the ray 3ϕ : Since $4 \cos^3 x = 3 \cos x + \cos 3x$, we have $4h_{n,1}^3 = 3h_{n,1} + h_{n,3}$, so

$$\begin{aligned} 3e^{-t}|E(te^{i\phi})| &= \left| 4 \sum_{n \geq 0} w_n s_n h_{n,1}^3 - \sum_{n \geq 0} w_n s_n h_{n,3} \right| \\ &= \left| 4 \sum_{n \geq 0} w_n s_n h_{n,1}^3 - \underbrace{e^{-t} \operatorname{Re} \left(e^{3i\theta} E(te^{3i\phi}) \right)}_{= o(1) \text{ by hypothesis}} \right|. \end{aligned}$$

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Cauchy–Schwarz:

$$3e^{-t}|E(te^{i\phi})| \leq 4 \limsup_{n \rightarrow \infty} |s_n| \left(\sum_{n \geq 0} w_n h_{n,1}^2 \right)^{1/2} \left(\sum_{n \geq 0} w_n h_{n,1}^4 \right)^{1/2} + o(1).$$

Proof of the Directional Estimate (3/3)

Remaining asymptotics: We can easily show that

$$\sum_{n \geq 0} w_n h_{n,1}^2 = \frac{1}{2} + o(1), \quad \sum_{n \geq 0} w_n h_{n,1}^4 = \frac{3}{8} + o(1).$$

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Remaining asymptotics: We can easily show that

$$\sum_{n \geq 0} w_n h_{n,1}^2 = \frac{1}{2} + o(1), \quad \sum_{n \geq 0} w_n h_{n,1}^4 = \frac{3}{8} + o(1).$$

$$\begin{aligned} \Rightarrow \quad 3e^{-t} |E(te^{i\phi})| &\leq 4 \limsup_{n \rightarrow \infty} |s_n| \sqrt{\frac{1}{2} \cdot \frac{3}{8}} + o(1) \\ &= \sqrt{3} \limsup_{n \rightarrow \infty} |s_n| + o(1). \end{aligned}$$

Proof of the Directional Estimate (3/3)

Remaining asymptotics: We can easily show that

$$\sum_{n \geq 0} w_n h_{n,1}^2 = \frac{1}{2} + o(1), \quad \sum_{n \geq 0} w_n h_{n,1}^4 = \frac{3}{8} + o(1).$$

$$\begin{aligned} \implies 3e^{-t} |E(te^{i\phi})| &\leq 4 \limsup_{n \rightarrow \infty} |s_n| \sqrt{\frac{1}{2} \cdot \frac{3}{8}} + o(1) \\ &= \sqrt{3} \limsup_{n \rightarrow \infty} |s_n| + o(1). \end{aligned}$$

Dividing by 3 gives the result. □

Reminder

The first lemma ensures that $G(te^{\pm 3i\phi}) = o(e^t)$ and $H(te^{\pm 3i\phi}) = o(e^t)$.

Lemma (B.; 2026+)

Let $0 < \phi < \pi/2$, let $(s_n)_{n \geq 0}$ be a bounded **real** sequence, and let $E(w) := \sum_{n \geq 0} s_n \frac{w^n}{n!}$. If

$$e^{-t} E(te^{\pm 3i\phi}) \longrightarrow 0 \quad (t \rightarrow \infty),$$

then

$$\limsup_{t \rightarrow \infty} e^{-t} |E(te^{\pm i\phi})| \leq \frac{1}{\sqrt{3}} \limsup_{n \rightarrow \infty} |s_n|.$$

Consequence for G and H

- $p = \sin \phi = \alpha/\beta$, $q = \cos \phi = 1/\beta$
- $x_n = a_n/\alpha^n$, $y_n = b_n/\beta^n$, $z_n = (-1)^n c_n/\beta^n$
- $G(w) = e^{qw} F(pw)$, $H(w) = e^{qw} F(-pw)$
- $B = \limsup |y_n|$, $C = \limsup |z_n|$

Applying the lemma with $E = G$ and $E = H$ yields

$$\limsup_{t \rightarrow \infty} e^{-t} |G(te^{\pm i\phi})| \leq \frac{1}{\sqrt{3}} \limsup_{n \rightarrow \infty} |y_n| = \frac{B}{\sqrt{3}},$$

$$\limsup_{t \rightarrow \infty} e^{-t} |H(te^{\pm i\phi})| \leq \frac{1}{\sqrt{3}} \limsup_{n \rightarrow \infty} |z_n| = \frac{C}{\sqrt{3}}.$$

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Remaining task

Transfer these estimates on the rays $\pm\phi$ to the saddle points of F .

Back to F

- $p = \sin \phi = \alpha/\beta$, $q = \cos \phi = 1/\beta$
- $x_n = a_n/\alpha^n$, $y_n = b_n/\beta^n$, $z_n = (-1)^n c_n/\beta^n$
- $G(w) = e^{qw} F(pw)$, $H(w) = e^{qw} F(-pw)$
- $B = \limsup |y_n|$, $C = \limsup |z_n|$

Let $\delta := \frac{\pi}{2} - \phi$ and

$$\Psi(w) := e^{-pw} F(-ipw).$$

Back to F

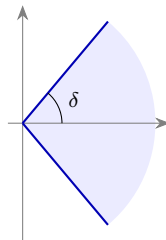
- $p = \sin \phi = \alpha/\beta$, $q = \cos \phi = 1/\beta$
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Let $\delta := \frac{\pi}{2} - \phi$ and

$$\Psi(w) := e^{-pw} F(-ipw).$$

Since $e^{i\delta} = p + iq$, the key identities give

$$\begin{aligned}\Psi(te^{i\delta}) &= e^{-t} G(te^{-i\phi}), \\ \Psi(te^{-i\delta}) &= e^{-t} H(te^{i\phi}).\end{aligned}$$



Back to F

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- $x_n = a_n/\alpha^n$, $y_n = b_n/\beta^n$, $z_n = (-1)^n c_n/\beta^n$
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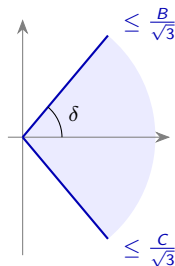
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$$\Psi(te^{i\delta}) = e^{-t} G(te^{-i\phi}),$$

$$\Psi(te^{-i\delta}) = e^{-t} H(te^{i\phi}).$$



So, asymptotically, $|\Psi| \leq B/\sqrt{3}$ on the upper ray and $|\Psi| \leq C/\sqrt{3}$ on the lower ray.

Back to F

- $p = \sin \phi = \alpha/\beta$, $q = \cos \phi = 1/\beta$
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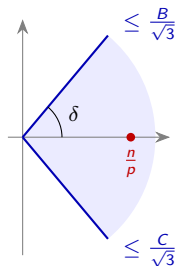
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So, asymptotically, $|\Psi| \leq B/\sqrt{3}$ on the upper ray and $|\Psi| \leq C/\sqrt{3}$ on the lower ray.

Moreover, for $r = n/p$, $|\Psi(re^{i\theta})| = e^{-n \cos \theta} |F(-ine^{i\theta})|$!

A Two-Constants Phragmén–Lindelöf Principle

Lemma

Let $0 < \delta < \pi/2$ and let f be holomorphic in $|\arg w| < \delta$, continuous on the boundary outside a compact set, and of exponential growth. If

$$\limsup_{r \rightarrow \infty} |f(re^{i\delta})| \leq M_+ \quad \text{and} \quad \limsup_{r \rightarrow \infty} |f(re^{-i\delta})| \leq M_-,$$

then, uniformly for θ in compact subsets of $(-\delta, \delta)$,

$$\limsup_{r \rightarrow \infty} |f(re^{i\theta})| \leq M_+^{\frac{\delta+\theta}{2\delta}} M_-^{\frac{\delta-\theta}{2\delta}}.$$

Conclusion of the Proof

- $p = \sin \phi = \alpha/\beta$, $q = \cos \phi = 1/\beta$
- $x_n = a_n/\alpha^n$, $y_n = b_n/\beta^n$, $z_n = (-1)^n c_n/\beta^n$
- $G(w) = e^{qw} F(pw)$, $H(w) = e^{qw} F(-pw)$
- $B = \limsup |y_n|$, $C = \limsup |z_n|$

Apply the lemma to Ψ with $M_+ = B/\sqrt{3}$ and $M_- = C/\sqrt{3}$:

$$\limsup_{r \rightarrow \infty} |\Psi(re^{i\theta})| \leq \frac{1}{\sqrt{3}} B^{\frac{\delta+\theta}{2\delta}} C^{\frac{\delta-\theta}{2\delta}}.$$

Conclusion of the Proof

- $p = \sin \phi = \alpha/\beta$, $q = \cos \phi = 1/\beta$
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Since $\eta_n \rightarrow 0$, taking $r = n/p$ gives

$$\sup_{|\theta| \leq \eta_n} e^{-n \cos \theta} |F(-ine^{i\theta})| \leq \frac{1}{\sqrt{3}} \sqrt{BC} + o(1).$$

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By symmetry, $|F(ine^{-i\theta})| = |F(-ine^{i\theta})|$. Hence

$$\sup_{|\theta| \leq \eta_n} e^{-n \cos \theta} (|F(-ine^{i\theta})| + |F(ine^{-i\theta})|) \leq \frac{2}{\sqrt{3}} \sqrt{BC} + o(1).$$

Conclusion of the Proof

- $p = \sin \phi = \alpha/\beta$, $q = \cos \phi = 1/\beta$
- $x_n = a_n/\alpha^n$, $y_n = b_n/\beta^n$, $z_n = (-1)^n c_n/\beta^n$
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$\Rightarrow$ The two points estimation lemma gives $\limsup_{n \rightarrow \infty} |x_n| \leq \frac{2}{\sqrt{3}} \sqrt{BC}$. $\square$

Putting Everything Together

$$b_n, c_n = O(\beta^n)$$

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binomial theorem

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estimate on the rays $\pm 3\phi$

$$G, H = o(e^t) \text{ on the rays } \pm 3\phi$$

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$$|G| \lesssim \frac{B}{\sqrt{3}} e^t \text{ and } |H| \lesssim \frac{C}{\sqrt{3}} e^t \text{ on the rays } \pm\phi$$

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two-constants PL applied to Ψ

$$|F| \lesssim \frac{1}{\sqrt{3}} \sqrt{BC} e^t \text{ on the rays } \pm \frac{\pi}{2}$$

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Cauchy's formula and the two points estimate

$$\limsup |x_n| \leq \frac{2}{\sqrt{3}} \sqrt{BC}$$

Back to the Main Result

Theorem (B.; 2026+)

Let $\beta > 1$, let $\alpha := \sqrt{\beta^2 - 1}$, and let (a_n) be a sequence of complex numbers. If $b_n, c_n = O(\beta^n)$, then

$$\limsup_{n \rightarrow \infty} \frac{|a_n|}{\alpha^n} \leq \frac{2}{\sqrt{3}} \left(\limsup_{n \rightarrow \infty} \frac{|b_n|}{\beta^n} \right)^{1/2} \left(\limsup_{n \rightarrow \infty} \frac{|c_n|}{\beta^n} \right)^{1/2},$$

and the constant $2/\sqrt{3}$ is optimal.

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