

A Finite-order Characterization of Entrywise Positivity Preservers

Ludovick Bouthat

University of Delaware

September 25, 2026

Notation

Definition

A real symmetric $n \times n$ matrix A is *positive semidefinite*, and we write $A \succeq 0$, if

$$\mathbf{x}^\top A \mathbf{x} \geq 0 \quad \text{for all } \mathbf{x} \in \mathbb{R}^n,$$

or equivalently, if all the eigenvalues of A are nonnegative.

For an interval $I \subseteq \mathbb{R}$, we denote by $\mathbb{P}_n(I)$ the set of $n \times n$ positive semidefinite matrices with entries in I .

Example

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \in \mathbb{P}_2((0, \infty)) \quad \text{but} \quad \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \notin \mathbb{P}_2((0, \infty)).$$

The Schur Product Theorem

Definition

The *Schur product* of two $n \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ is the entrywise product

$$A \circ B := [a_{ij}b_{ij}]_{i,j=1}^n.$$

The Schur Product Theorem

Definition

The *Schur product* of two $n \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ is the entrywise product

$$A \circ B := [a_{ij}b_{ij}]_{i,j=1}^n.$$

Theorem (Schur; 1911)

If $A, B \in \mathbb{P}_n(\mathbb{R})$, then $A \circ B \in \mathbb{P}_n(\mathbb{R})$.

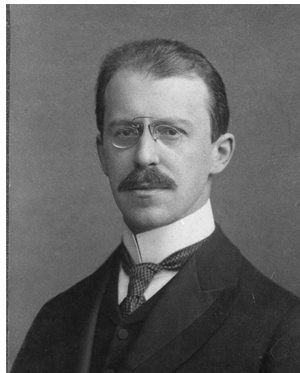


Figure: Issai Schur

Consequences of Schur's Theorem

- By induction, if $A \succeq 0$, then $A^{\circ k} := [a_{ij}^k]_{i,j=1}^n \succeq 0$ for every $k \in \mathbb{N}$.

Consequences of Schur's Theorem

- By induction, if $A \succeq 0$, then $A^{\circ k} := [a_{ij}^k]_{i,j=1}^n \succeq 0$ for every $k \in \mathbb{N}$.
- Since sums of positive semidefinite matrices are positive semidefinite,

$$\sum_{k \geq 0} c_k A^{\circ k} \succeq 0 \quad \text{whenever } c_k \geq 0.$$

Consequences of Schur's Theorem

- By induction, if $A \succeq 0$, then $A^{\circ k} := [a_{ij}^k]_{i,j=1}^n \succeq 0$ for every $k \in \mathbb{N}$.
- Since sums of positive semidefinite matrices are positive semidefinite,

$$\sum_{k \geq 0} c_k A^{\circ k} \succeq 0 \quad \text{whenever } c_k \geq 0.$$

Definition

Given $f : I \rightarrow \mathbb{R}$ and $A = [a_{ij}]$ with entries in I , let $f[A] := [f(a_{ij})]_{i,j=1}^n$. We say that f *preserves positivity* on $\mathbb{P}_n(I)$ if $f[A] \succeq 0$ for every $A \in \mathbb{P}_n(I)$.

Consequences of Schur's Theorem

- By induction, if $A \succeq 0$, then $A^{\circ k} := [a_{ij}^k]_{i,j=1}^n \succeq 0$ for every $k \in \mathbb{N}$.
- Since sums of positive semidefinite matrices are positive semidefinite,

$$\sum_{k \geq 0} c_k A^{\circ k} \succeq 0 \quad \text{whenever } c_k \geq 0.$$

Definition

Given $f : I \rightarrow \mathbb{R}$ and $A = [a_{ij}]$ with entries in I , let $f[A] := [f(a_{ij})]_{i,j=1}^n$. We say that f *preserves positivity* on $\mathbb{P}_n(I)$ if $f[A] \succeq 0$ for every $A \in \mathbb{P}_n(I)$.

Corollary (Schur; 1911)

If $f(x) = \sum_{k \geq 0} c_k x^k$ on I with $c_k \geq 0$ (i.e., f is absolutely monotonic), then $f[A] \succeq 0$ preserves positivity on $\mathbb{P}_n(I)$ for every $n \geq 1$.

Every Dimension: a Complete Answer

Theorem (Schoenberg, 1942; Rudin, 1959; Vasudeva, 1979)

Let $I = (0, \rho)$ with $0 < \rho \leq \infty$ and let $f : I \rightarrow \mathbb{R}$. The following are equivalent:

- 1 f preserves positivity on $\mathbb{P}_n(I)$ for **every** $n \geq 1$;
- 2 $f(x) = \sum_{k \geq 0} c_k x^k$ on I , with $c_k \geq 0$ for all $k \geq 0$.

What About a Fixed Dimension?

Question (Pólya and Szegő; 1925)

Fix $n \geq 1$. Which functions $f : I \rightarrow \mathbb{R}$ preserve positivity on $\mathbb{P}_n(I)$?

What About a Fixed Dimension?

Question (Pólya and Szegő; 1925)

Fix $n \geq 1$. Which functions $f : I \rightarrow \mathbb{R}$ preserve positivity on $\mathbb{P}_n(I)$?

- From now on, $I = (0, \rho)$, where $0 < \rho \leq \infty$.

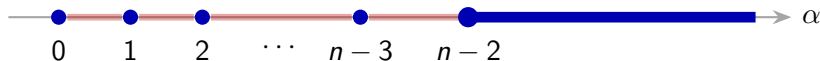
Fractional Hadamard Powers

Theorem (FitzGerald, Horn; 1977)

Let $n \geq 2$ and $\alpha \in \mathbb{R}$. Then x^α preserves positivity on $\mathbb{P}_n((0, \infty))$ if and only if

$$\alpha \in \mathbb{N}_0 \cup [n-2, \infty),$$

where $\mathbb{N}_0 := \{0, 1, 2, \dots\}$.



The Horn–Loewner Necessary Condition

Theorem (Horn, Loewner; 1969)

Let $n \geq 2$ and $f \in C^{n-1}(I)$. If f preserves positivity on $\mathbb{P}_n(I)$, then

$$f^{(k)}(x) \geq 0, \quad x \in I, \quad 0 \leq k \leq n-1.$$

The Horn–Loewner Necessary Condition

Theorem (Horn, Loewner; 1969)

Let $n \geq 2$ and $f \in C^{n-1}(I)$. If f preserves positivity on $\mathbb{P}_n(I)$, then

$$f^{(k)}(x) \geq 0, \quad x \in I, \quad 0 \leq k \leq n-1.$$

FitzGerald–Horn

We have $(x^\alpha)^{(k)} = \alpha(\alpha-1)\cdots(\alpha-k+1)x^{\alpha-k}$. If $\alpha \notin \mathbb{N}_0 \cup [n-2, \infty)$, one of these derivatives is negative for some $k \leq n-1$.

The Horn–Loewner Necessary Condition

Theorem (Horn, Loewner; 1969)

Let $n \geq 2$ and $f \in C^{n-1}(I)$. If f preserves positivity on $\mathbb{P}_n(I)$, then

$$f^{(k)}(x) \geq 0, \quad x \in I, \quad 0 \leq k \leq n-1.$$

FitzGerald–Horn

We have $(x^\alpha)^{(k)} = \alpha(\alpha-1)\cdots(\alpha-k+1)x^{\alpha-k}$. If $\alpha \notin \mathbb{N}_0 \cup [n-2, \infty)$, one of these derivatives is negative for some $k \leq n-1$.

Power series

If $f(x) = \sum_{k \geq 0} c_k x^k$ preserves positivity on $\mathbb{P}_n(I)$, then $c_0, \dots, c_{n-1} \geq 0$, but there is **no sign restriction** on $c_n, c_{n+1}, \dots$

The Case $n = 2$

Theorem (Vasudeva; 1979)

A function $f : (0, \infty) \rightarrow \mathbb{R}$ preserves positivity on $\mathbb{P}_2((0, \infty))$ if and only if f is nonnegative, nondecreasing, and multiplicatively midconvex:

$$f(\sqrt{xy})^2 \leq f(x)f(y), \quad x, y > 0.$$

Horn–Loewner is not sufficient

The function $f(x) = \frac{x}{1+x}$ satisfies $f, f' \geq 0$, but

$$A = \begin{bmatrix} 1/4 & 1 \\ 1 & 4 \end{bmatrix} \succeq 0 \quad \text{and} \quad \det f[A] = \det \begin{bmatrix} 1/5 & 1/2 \\ 1/2 & 4/5 \end{bmatrix} = -\frac{9}{100}.$$

Polynomials with a Negative Coefficient

Theorem (Belton, Guillot, Khare, Putinar; 2016)

Fix $n \geq 2$, an integer $M \geq n$, $0 < \rho < \infty$, and $c_0, \dots, c_{n-1} > 0$. Then

$$f_\gamma(x) = c_0 + c_1x + \dots + c_{n-1}x^{n-1} - \gamma x^M$$

preserves positivity on $\mathbb{P}_n((0, \rho))$ if and only if $\gamma \leq C^{-1}$, where

$$C := \sum_{j=0}^{n-1} \binom{M}{j}^2 \binom{M-j-1}{n-j-1}^2 \frac{\rho^{M-j}}{c_j}.$$

Polynomials with a Negative Coefficient

Theorem (Belton, Guillot, Khare, Putinar; 2016)

Fix $n \geq 2$, an integer $M \geq n$, $0 < \rho < \infty$, and $c_0, \dots, c_{n-1} > 0$. Then

$$f_\gamma(x) = c_0 + c_1x + \dots + c_{n-1}x^{n-1} - \gamma x^M$$

preserves positivity on $\mathbb{P}_n((0, \rho))$ if and only if $\gamma \leq C^{-1}$, where

$$C := \sum_{j=0}^{n-1} \binom{M}{j}^2 \binom{M-j-1}{n-j-1}^2 \frac{\rho^{M-j}}{c_j}.$$

Example

$1 + x - \frac{1}{5}x^2$ preserves positivity on $\mathbb{P}_2((0, 1))$, and the constant $\frac{1}{5}$ is sharp.

$\implies$ Preservers in fixed dimension can have *negative* coefficients!

The Extension Principle

- A few years later, Khare and Tao extended this to sums of real powers x^{α_j} , with explicit sharp thresholds. More importantly for us, they proved:

Theorem (Khare, Tao; 2021)

Let $n \geq 2$, $I = (0, \rho)$, and $q \in C^1(I)$. Suppose that

- (a) $q[R] \succeq 0$ for all $R \in \mathbb{P}_n^1(I)$, the rank-one matrices of $\mathbb{P}_n(I)$;
- (b) $q'[C] \succeq 0$ for all $C \in \mathbb{P}_{n-1}(I)$.

Then $q[A] \succeq 0$ for all $A \in \mathbb{P}_n(I)$.

What We Knew (Early September 2026)

Dimension	Status	
$n = 1$	solved	trivial: $f \geq 0$
$n = 2$	solved	Vasudeva (1979)
$n \geq 3$	open	necessary conditions (Horn–Loewner) + special families (powers, BGKP, Khare–Tao)
all n	solved	Schoenberg, Rudin, Vasudeva

What We Knew (Early September 2026)

Dimension	Status	
$n = 1$	solved	trivial: $f \geq 0$
$n = 2$	solved	Vasudeva (1979)
$n \geq 3$	open	necessary conditions (Horn–Loewner) + special families (powers, BGKP, Khare–Tao)
all n	solved	Schoenberg, Rudin, Vasudeva

Even for $n = 3$: no characterization, and **not even a conjecture!**

The Euler–Hankel Matrix

Definition

The *Euler derivative* is $\mathcal{E} = x \frac{d}{dx}$. The *Euler–Hankel matrix* of $f \in C^{2n-2}(I)$ is

$$\mathcal{H}_n(f; x) := [\mathcal{E}^{i+j}f(x)]_{i,j=0}^{n-1} = \begin{bmatrix} f & \mathcal{E}f & \dots & \mathcal{E}^{n-1}f \\ \mathcal{E}f & \mathcal{E}^2f & \dots & \mathcal{E}^nf \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{E}^{n-1}f & \mathcal{E}^nf & \dots & \mathcal{E}^{2n-2}f \end{bmatrix} (x).$$

Example

Since $\mathcal{E}f = xf'$ and $\mathcal{E}^2f = xf' + x^2f''$, we have

$$\mathcal{H}_2(f; x) = \begin{bmatrix} f(x) & xf'(x) \\ xf'(x) & xf'(x) + x^2f''(x) \end{bmatrix}.$$

A Complete Characterization

Theorem (B., Guillot; 2026+)

Fix $n \geq 1$, let $I = (0, \rho)$ with $0 < \rho \leq \infty$, and let $f \in C^{2n-2}(I)$. Then f preserves positivity on $\mathbb{P}_n(I)$ if and only if, for every $x \in I$,

$$\underbrace{f^{(k)}(x) \geq 0 \quad (0 \leq k \leq n-1)}_{\text{Horn-Loewner}} \quad \text{and} \quad \underbrace{\mathcal{H}_n(f; x) \succeq 0}_{\text{Euler-Hankel}}.$$

Comments on the Main Theorem

- **Local:** a global condition (for all $A \in \mathbb{P}_n(I)$) becomes finitely many *pointwise* differential inequalities, since

$$\mathcal{E}^k f(x) = \sum_{\ell=0}^k \left\{ \begin{matrix} k \\ \ell \end{matrix} \right\} x^\ell f^{(\ell)}(x) \quad (\text{Stirling numbers}).$$

Comments on the Main Theorem

- **Local:** a global condition (for all $A \in \mathbb{P}_n(I)$) becomes finitely many *pointwise* differential inequalities, since

$$\mathcal{E}^k f(x) = \sum_{\ell=0}^k \left\{ \begin{matrix} k \\ \ell \end{matrix} \right\} x^\ell f^{(\ell)}(x) \quad (\text{Stirling numbers}).$$

- **Complete:** no assumption on the form of f (no power series, no sign pattern), and it holds for every n .

Comments on the Main Theorem

- **Local:** a global condition (for all $A \in \mathbb{P}_n(I)$) becomes finitely many *pointwise* differential inequalities, since

$$\mathcal{E}^k f(x) = \sum_{\ell=0}^k \left\{ \begin{matrix} k \\ \ell \end{matrix} \right\} x^\ell f^{(\ell)}(x) \quad (\text{Stirling numbers}).$$

- **Complete:** no assumption on the form of f (no power series, no sign pattern), and it holds for every n .
- **Regularity:** C^{2n-2} is exactly what is needed to write down $\mathcal{H}_n(f; x)$. In fact, every preserver is automatically in $C^{2n-4}(I)$ when $n \geq 2$, and the smoothness assumption can be removed using distributions.

Strategy: Reduction to Rank One

Theorem (Khare, Tao; 2021)

Let $n \geq 2$, $I = (0, \rho)$, and $q \in C^1(I)$. Suppose that

- (a) $q[R] \succeq 0$ for all $R \in \mathbb{P}_n^1(I)$, the rank-one matrices of $\mathbb{P}_n(I)$;
- (b) $q'[C] \succeq 0$ for all $C \in \mathbb{P}_{n-1}(I)$.

Then $q[A] \succeq 0$ for all $A \in \mathbb{P}_n(I)$.

Strategy: Reduction to Rank One

Theorem (Khare, Tao; 2021)

Let $n \geq 2$, $I = (0, \rho)$, and $q \in C^1(I)$. Suppose that

- (a) $q[R] \succeq 0$ for all $R \in \mathbb{P}_n^1(I)$, the rank-one matrices of $\mathbb{P}_n(I)$;
- (b) $q'[C] \succeq 0$ for all $C \in \mathbb{P}_{n-1}(I)$.

Then $q[A] \succeq 0$ for all $A \in \mathbb{P}_n(I)$.

Bad news

Rank-one preservers were not understood either. . .

The (Mandatory) AI Slide

- While exploring proof strategies, an AI model (ChatGPT) pointed us to a 1968 result of Karlin, hidden in his book *Total Positivity*.

The (Mandatory) AI Slide

- While exploring proof strategies, an AI model (ChatGPT) pointed us to a 1968 result of Karlin, hidden in his book *Total Positivity*.

Lemma (Karlin; 1968)

Let $X \subseteq \mathbb{R}$ be an open interval and $q \in C^{2m-2}(X + X)$. Then

$$[q(s_i + s_j)]_{i,j=1}^m \succeq 0 \quad \forall s_i \in X \quad \Longleftrightarrow \quad \mathbf{H}_m(q; t) \succeq 0 \quad \forall t \in X + X,$$

where $\mathbf{H}_m(q; t) := [q^{(i+j)}(t)]_{i,j=0}^{m-1}$.

From Rank One to Hankel Matrices

- Let $g(t) := f(e^t)$. Then

$$g(x + y) = f(e^x \cdot e^y) \quad \& \quad g^{(k)}(t) = (\mathcal{E}^k f)(e^t)$$

From Rank One to Hankel Matrices

- Let $g(t) := f(e^t)$. Then

$$g(x + y) = f(e^x \cdot e^y) \quad \& \quad g^{(k)}(t) = (\mathcal{E}^k f)(e^t)$$

If $R \in \mathbb{P}_n^1(I)$, then $R = \mathbf{u}\mathbf{u}^\top$ with $u_i > 0$, i.e., $r_{ij} = u_i u_j$. With $s_i := \log u_i$,

$$f[R] = [f(u_i u_j)]_{i,j=1}^n = [f(e^{s_i + s_j})]_{i,j=1}^n = [g(s_i + s_j)]_{i,j=1}^n.$$

From Rank One to Hankel Matrices

- Let $g(t) := f(e^t)$. Then

$$g(x + y) = f(e^x \cdot e^y) \quad \& \quad g^{(k)}(t) = (\mathcal{E}^k f)(e^t)$$

If $R \in \mathbb{P}_n^1(I)$, then $R = \mathbf{u}\mathbf{u}^\top$ with $u_i > 0$, i.e., $r_{ij} = u_i u_j$. With $s_i := \log u_i$,

$$f[R] = [f(u_i u_j)]_{i,j=1}^n = [f(e^{s_i + s_j})]_{i,j=1}^n = [g(s_i + s_j)]_{i,j=1}^n.$$

Therefore, by Karlin's lemma,

$$\begin{aligned} f[R] \succeq 0 \quad \forall R \in \mathbb{P}_n^1(I) &\iff [g^{(i+j)}(t)]_{i,j=0}^{n-1} \succeq 0 \quad \forall t \in \log I \\ &\iff \mathcal{H}_n(f; x) \succeq 0 \quad \forall x \in I. \end{aligned}$$

Rank-One Preservers

Proposition (B., Guillot; 2026+)

Let $f \in C^{2n-2}(I)$. Then $f[R] \succeq 0$ for all $R \in \mathbb{P}_n^1(I)$ if and only if

$$\mathcal{H}_n(f; x) \succeq 0 \quad \text{for all } x \in I.$$

Rank-One Preservers

Proposition (B., Guillot; 2026+)

Let $f \in C^{2n-2}(I)$. Then $f[R] \succeq 0$ for all $R \in \mathbb{P}_n^1(I)$ if and only if

$$\mathcal{H}_n(f; x) \succeq 0 \quad \text{for all } x \in I.$$

Corollary (B., Guillot; 2026+)

If $f \in C^{2n-2}(I)$ preserves positivity on $\mathbb{P}_n(I)$, then for all $x \in I$,

$$\underbrace{f^{(k)}(x) \geq 0 \quad (0 \leq k \leq n-1)}_{\text{Horn-Loewner}} \quad \text{and} \quad \underbrace{\mathcal{H}_n(f; x) \succeq 0}_{\text{rank-one case}}.$$

Rank-One Preservers

Proposition (B., Guillot; 2026+)

Let $f \in C^{2n-2}(I)$. Then $f[R] \succeq 0$ for all $R \in \mathbb{P}_n^1(I)$ if and only if

$$\mathcal{H}_n(f; x) \succeq 0 \quad \text{for all } x \in I.$$

Corollary (B., Guillot; 2026+)

If $f \in C^{2n-2}(I)$ preserves positivity on $\mathbb{P}_n(I)$, then for all $x \in I$,

$$\underbrace{f^{(k)}(x) \geq 0 \quad (0 \leq k \leq n-1)}_{\text{Horn-Loewner}} \quad \text{and} \quad \underbrace{\mathcal{H}_n(f; x) \succeq 0}_{\text{rank-one case}}.$$

Claim

These conditions are also **sufficient**.

The Induction

- Assume that $f^{(k)} \geq 0$ ($0 \leq k \leq n-1$) and $\mathcal{H}_n(f; \cdot) \succeq 0$ on I . By Khare–Tao, it suffices to show that:
 - (a) f preserves positivity on $\mathbb{P}_n^1(I)$: ✓ by the rank-one lemma;
 - (b) f' preserves positivity on $\mathbb{P}_{n-1}(I)$.

The Induction

- Assume that $f^{(k)} \geq 0$ ($0 \leq k \leq n-1$) and $\mathcal{H}_n(f; \cdot) \succeq 0$ on I . By Khare–Tao, it suffices to show that:
 - (a) f preserves positivity on $\mathbb{P}_n^1(I)$: ✓ by the rank-one lemma;
 - (b) f' preserves positivity on $\mathbb{P}_{n-1}(I)$.

By the induction hypothesis, (b) holds as soon as f' satisfies the conditions in dimension $n-1$:

- $(f')^{(k)} = f^{(k+1)} \geq 0$ for $0 \leq k \leq n-2$: ✓
- $\mathcal{H}_{n-1}(f'; x) \succeq 0$ for all $x \in I$: ?

The Induction

- Assume that $f^{(k)} \geq 0$ ($0 \leq k \leq n-1$) and $\mathcal{H}_n(f; \cdot) \succeq 0$ on I . By Khare–Tao, it suffices to show that:
 - f preserves positivity on $\mathbb{P}_n^1(I)$: ✓ by the rank-one lemma;
 - f' preserves positivity on $\mathbb{P}_{n-1}(I)$.

By the induction hypothesis, (b) holds as soon as f' satisfies the conditions in dimension $n-1$:

- $(f')^{(k)} = f^{(k+1)} \geq 0$ for $0 \leq k \leq n-2$: ✓
- $\mathcal{H}_{n-1}(f'; x) \succeq 0$ for all $x \in I$: ?

Missing step

$$\mathcal{H}_n(f; x) \succeq 0 \quad \xRightarrow{?} \quad \mathcal{H}_{n-1}(f'; x) \succeq 0.$$

The Missing Step

Proposition (B., Guillot; 2026+)

Let $m \geq 2$ and let $f \in C^{2m}(I)$ be such that $f, f', f'' \geq 0$ on I . If

$$\mathcal{H}_{m+1}(f; x) \succeq 0 \quad \text{for all } x \in I,$$

then

$$\mathcal{H}_m(f'; x) \succeq 0 \quad \text{for all } x \in I.$$

The Missing Step

Proposition (B., Guillot; 2026+)

Let $m \geq 2$ and let $f \in C^{2m}(I)$ be such that $f, f', f'' \geq 0$ on I . If

$$\mathcal{H}_{m+1}(f; x) \succeq 0 \quad \text{for all } x \in I,$$

then

$$\mathcal{H}_m(f'; x) \succeq 0 \quad \text{for all } x \in I.$$

Remark

With $m = n - 1 \geq 2$, this is exactly the missing step: the conditions $f, f', f'' \geq 0$ are part of Horn–Loewner when $n \geq 3$.

Proof of the Descent Lemma (1/2)

- Let $g(t) := f(e^t)$ on $J := \log I = (-\infty, \log \rho)$, and $X := (-\infty, \frac{1}{2} \log \rho)$.

Proof of the Descent Lemma (1/2)

- Let $g(t) := f(e^t)$ on $J := \log I = (-\infty, \log \rho)$, and $X := (-\infty, \frac{1}{2} \log \rho)$.

Step 1: The lower-right $m \times m$ block of $\mathcal{H}_{m+1}(f; e^t) = [g^{(i+j)}(t)]_{i,j=0}^m$ is

$$[g^{(i+j+2)}(t)]_{i,j=0}^{m-1} = \mathbf{H}_m(g''; t) \succeq 0 \xrightarrow{\text{Karlin}} [g''(s_i + s_j)]_{i,j=1}^m \succeq 0 \quad \forall s_i \in X.$$

Proof of the Descent Lemma (1/2)

- Let $g(t) := f(e^t)$ on $J := \log I = (-\infty, \log \rho)$, and $X := (-\infty, \frac{1}{2} \log \rho)$.

Step 1: The lower-right $m \times m$ block of $\mathcal{H}_{m+1}(f; e^t) = [g^{(i+j)}(t)]_{i,j=0}^m$ is

$$[g^{(i+j+2)}(t)]_{i,j=0}^{m-1} = \mathbf{H}_m(g''; t) \succeq 0 \xrightarrow{\text{Karlin}} [g''(s_i + s_j)]_{i,j=1}^m \succeq 0 \quad \forall s_i \in X.$$

Step 2: We have $g'(t) = e^t f'(e^t) \geq 0$ and $g''(t) - g'(t) = e^{2t} f''(e^t) \geq 0$. Thus g' is nondecreasing and $\ell := \lim_{t \rightarrow -\infty} g'(t) \geq 0$ exists.

Proof of the Descent Lemma (1/2)

- Let $g(t) := f(e^t)$ on $J := \log I = (-\infty, \log \rho)$, and $X := (-\infty, \frac{1}{2} \log \rho)$.

Step 1: The lower-right $m \times m$ block of $\mathcal{H}_{m+1}(f; e^t) = [g^{(i+j)}(t)]_{i,j=0}^m$ is

$$[g^{(i+j+2)}(t)]_{i,j=0}^{m-1} = \mathbf{H}_m(g''; t) \succeq 0 \xrightarrow{\text{Karlin}} [g''(s_i + s_j)]_{i,j=1}^m \succeq 0 \quad \forall s_i \in X.$$

Step 2: We have $g'(t) = e^t f'(e^t) \geq 0$ and $g''(t) - g'(t) = e^{2t} f''(e^t) \geq 0$. Thus g' is nondecreasing and $\ell := \lim_{t \rightarrow -\infty} g'(t) \geq 0$ exists.

If $\ell > 0$, then $g(t) = g(T) - \int_t^T g'(u) \, du \leq g(T) - \ell(T - t) \rightarrow -\infty$ as $t \rightarrow -\infty$, contradicting $g \geq 0$. Hence $\ell = 0$, and

$$g'(x) = \int_{-\infty}^x g''(u) \, du = \int_0^\infty g''(x - \tau) \, d\tau.$$

Proof of the Descent Lemma (2/2)

Step 3: For all $s_1, \dots, s_m \in X$,

$$[g'(s_i + s_j)]_{i,j=1}^m = \int_0^\infty \left[g''((s_i - \frac{\tau}{2}) + (s_j - \frac{\tau}{2})) \right]_{i,j=1}^m d\tau \succeq 0,$$

since $s_i - \frac{\tau}{2} \in X$. Here, it is crucial that J is unbounded to the left!

Proof of the Descent Lemma (2/2)

Step 3: For all $s_1, \dots, s_m \in X$,

$$[g'(s_i + s_j)]_{i,j=1}^m = \int_0^\infty \left[g''((s_i - \frac{\tau}{2}) + (s_j - \frac{\tau}{2})) \right]_{i,j=1}^m d\tau \succeq 0,$$

since $s_i - \frac{\tau}{2} \in X$. Here, it is crucial that J is unbounded to the left!

Step 4: Let $h(t) := e^{-t}g'(t) = f'(e^t)$ and $D := \text{diag}(e^{-s_1}, \dots, e^{-s_m})$. Then

$$[h(s_i + s_j)]_{i,j=1}^m = D [g'(s_i + s_j)]_{i,j=1}^m D \succeq 0.$$

Proof of the Descent Lemma (2/2)

Step 3: For all $s_1, \dots, s_m \in X$,

$$[g'(s_i + s_j)]_{i,j=1}^m = \int_0^\infty \left[g''((s_i - \frac{\tau}{2}) + (s_j - \frac{\tau}{2})) \right]_{i,j=1}^m d\tau \succeq 0,$$

since $s_i - \frac{\tau}{2} \in X$. **Here, it is crucial that J is unbounded to the left!**

Step 4: Let $h(t) := e^{-t}g'(t) = f'(e^t)$ and $D := \text{diag}(e^{-s_1}, \dots, e^{-s_m})$. Then

$$[h(s_i + s_j)]_{i,j=1}^m = D [g'(s_i + s_j)]_{i,j=1}^m D \succeq 0.$$

Step 5: By Karlin, $\mathbf{H}_m(h; t) \succeq 0$. Since $h^{(k)}(t) = (\mathcal{E}^k f')(e^t)$, this is exactly

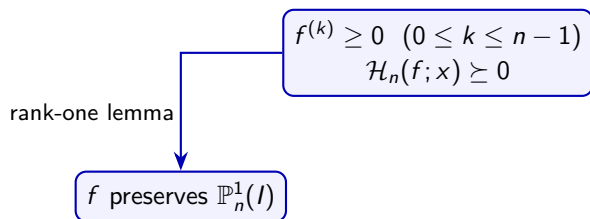
$$\mathcal{H}_m(f'; x) \succeq 0 \quad \text{for all } x \in I.$$

□

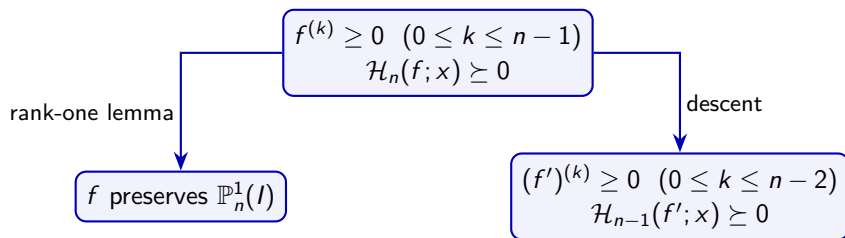
Putting Everything Together

$$\begin{aligned} f^{(k)} &\geq 0 \quad (0 \leq k \leq n-1) \\ \mathcal{H}_n(f; x) &\succeq 0 \end{aligned}$$

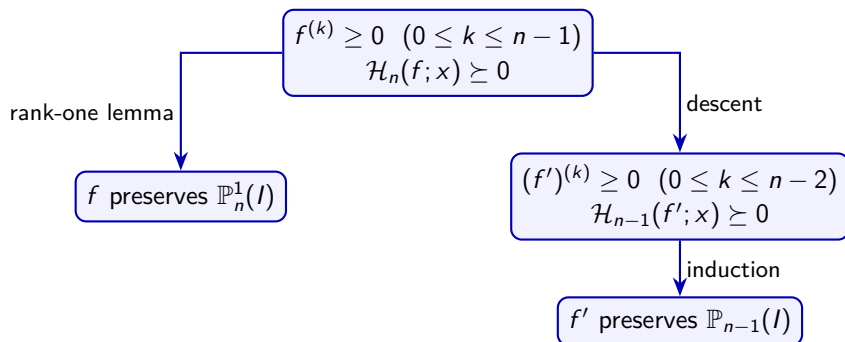
Putting Everything Together



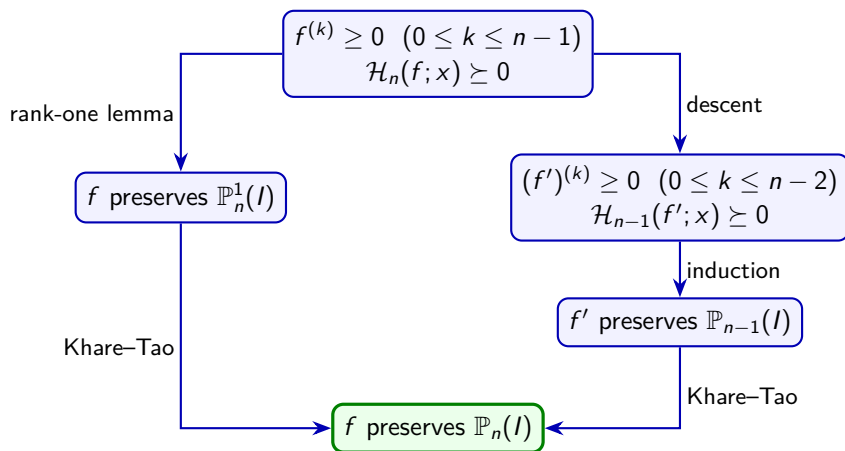
Putting Everything Together



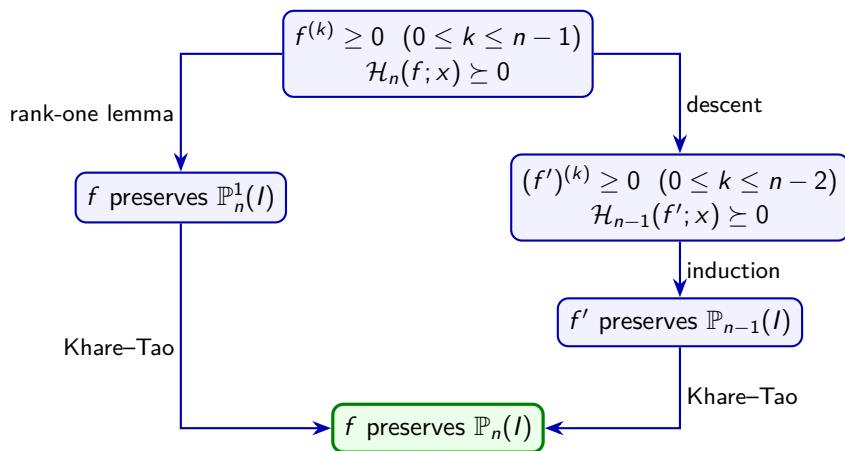
Putting Everything Together



Putting Everything Together



Putting Everything Together



The converse follows from Horn–Loewner and the rank-one lemma.

Back to the Main Result

Theorem (B., Guillot; 2026+)

Let $f \in C^{2n-2}(I)$. Then f preserves positivity on $\mathbb{P}_n(I)$ if and only if $f^{(k)} \geq 0$ for $0 \leq k \leq n-1$ and $\mathcal{H}_n(f; x) \succeq 0$ for all $x \in I$.

Back to the Main Result

Theorem (B., Guillot; 2026+)

Let $f \in C^{2n-2}(I)$. Then f preserves positivity on $\mathbb{P}_n(I)$ if and only if $f^{(k)} \geq 0$ for $0 \leq k \leq n-1$ and $\mathcal{H}_n(f; x) \succeq 0$ for all $x \in I$.

Some consequences (see the paper)

- Same result for arbitrary f , using distributional derivatives.
- Recovers FitzGerald–Horn, Vasudeva, BGKP, and Khare–Tao.
- Classification of $\sum_j c_j x^{\alpha_j} + c' x^M$ with *arbitrary* real coefficients.
- Sharp thresholds for mixtures of powers $\int x^\alpha d\mu(\alpha) - cx^M$.
- Optimal constants K in $g[A] \preceq K h[A]$ under finite regularity.
- No finite set of test matrices with entries inside I can detect positivity preservation (answering a question of Khare–Tao).

References I



A. Belton, D. Guillot, A. Khare, and M. Putinar. Matrix positivity preservers in fixed dimension. I. *Adv. Math.*, 298:325–368, 2016.



L. Bouthat and D. Guillot. A finite-order characterization of entrywise positivity preservers. Preprint, 2026.



C. H. FitzGerald and R. A. Horn. On fractional Hadamard powers of positive definite matrices. *J. Math. Anal. Appl.*, 61(3):633–642, 1977.



R. A. Horn. The theory of infinitely divisible matrices and kernels. *Trans. Amer. Math. Soc.*, 136:269–286, 1969.



S. Karlin. *Total Positivity. Vol. I*. Stanford University Press, Stanford, CA, 1968.



A. Khare. *Matrix Analysis and Entrywise Positivity Preservers*. London Math. Soc. Lecture Note Ser. 471, Cambridge University Press, 2022.



A. Khare and T. Tao. On the sign patterns of entrywise positivity preservers in fixed dimension. *Amer. J. Math.*, 143(6):1863–1929, 2021.

References II



W. Rudin. Positive definite sequences and absolutely monotonic functions. *Duke Math. J.*, 26(4):617–622, 1959.



I. J. Schoenberg. Positive definite functions on spheres. *Duke Math. J.*, 9(1):96–108, 1942.



I. Schur. Bemerkungen zur Theorie der beschränkten Bilinearformen mit unendlich vielen Veränderlichen. *J. Reine Angew. Math.*, 140:1–28, 1911.



H. L. Vasudeva. Positive definite matrices and absolutely monotonic functions. *Indian J. Pure Appl. Math.*, 10(7):854–858, 1979.

One Generalization

Theorem (B., Guillot; 2026+)

Fix $n \geq 1$, $0 < \rho \leq \infty$ and $I = (0, \rho)$. Let $g, h \in C^{2n-2}(I)$, and suppose that h preserves positivity on $\mathbb{P}_n(I)$. Define

$$\gamma_n(h, g; x) := \lambda_{\max} \left(\mathcal{H}_n(h; x)^{-1/2} \mathcal{H}_n(g; x) \mathcal{H}_n(h; x)^{-1/2} \right)$$

and

$$\mathcal{K}_n(h, g; I) := \sup_{x \in I} \max \left\{ 0, \max_{0 \leq r \leq n-1} \frac{g^{(r)}(x)}{h^{(r)}(x)}, \gamma_n(h, g; x) \right\}.$$

Then, for every $K \geq 0$, the following assertions are equivalent:

- ① $g[A] \preceq Kh[A]$ for every $A \in \mathbb{P}_n(I)$,
- ② $K \geq \mathcal{K}_n(h, g; I)$.

Arbitrary Functions

Theorem (B., Guillot; 2026+)

Fix $n \geq 2$, let $f : I \rightarrow \mathbb{R}$, and let $g(t) := f(e^t)$ on $J = \log I$. Then f preserves positivity on $\mathbb{P}_n(I)$ if and only if f is continuous, $D^k f \geq 0$ in $\mathcal{D}'(I)$ for $0 \leq k \leq n-1$, and

$$\left[\langle D^{i+j} g, \psi \rangle \right]_{i,j=0}^{n-1} \succeq 0 \quad \text{for every } \psi \in C_c^\infty(J), \psi \geq 0.$$

Arbitrary Functions

Theorem (B., Guillot; 2026+)

Fix $n \geq 2$, let $f : I \rightarrow \mathbb{R}$, and let $g(t) := f(e^t)$ on $J = \log I$. Then f preserves positivity on $\mathbb{P}_n(I)$ if and only if f is continuous, $D^k f \geq 0$ in $\mathcal{D}'(I)$ for $0 \leq k \leq n-1$, and

$$\left[\langle D^{i+j} g, \psi \rangle \right]_{i,j=0}^{n-1} \succeq 0 \quad \text{for every } \psi \in C_c^\infty(J), \psi \geq 0.$$

Idea

The multiplicative regularization $f_\varepsilon(x) := \int_0^\varepsilon \eta_\varepsilon(s) f(e^{-s}x) ds$ still preserves positivity, since $e^{-s}A \in \mathbb{P}_n(I)$ whenever $A \in \mathbb{P}_n(I)$.

$\implies$ Every preserver belongs to $C^{2n-4}(I)$.