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Number of orbits of k -subsets of permutations. (English. English summary)

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Let S_n denote the symmetric group of degree n and let $T(n, k)$ be the number of equivalence classes of k -element subsets of S_n , where two subsets $X, Y \subseteq S_n$ are declared equivalent if there exist permutations $g_1, g_2 \in S_n$ such that

$$Y = g_1 X g_2,$$

where multiplication is understood elementwise. Thus, $T(n, k)$ is the number of orbits of the two-sided multiplication action of $S_n \times S_n$ on the family of k -element subsets of S_n . The cases $k = 0, 1, 2$ are elementary:

$$T(n, 0) = T(n, 1) = 1 \quad \text{and} \quad T(n, 2) = p(n) - 1,$$

where $p(n)$ denotes the number of integer partitions of n . One also has the symmetry $T(n, k) = T(n, n! - k)$. While exact values are known for $n \leq 6$, computing $T(n, k)$ for even moderately large n appears to be out of reach.

The problem is motivated in part by the study of Erdős matrices, namely, doubly stochastic matrices saturating the following inequality of Marcus and Ree:

$$\sum_{i,j} a_{ij}^2 \leq \max_{\sigma \in S_n} \sum_i a_{i, \sigma(i)}.$$

The connection with the present paper stems from an observation of Tripathi: every Erdős matrix belongs to the convex hull of a linearly independent collection of permutation matrices, and each such convex hull contains at most one Erdős matrix. Consequently, understanding subsets of permutation matrices modulo two-sided multiplication is relevant to the classification of Erdős matrices up to permutation equivalence.

The principal result of the paper is the following uniform asymptotic formula:

$$T(n, k) = \frac{1}{(n!)^2} \binom{n!}{k} (1 + O(n^{-2})), \quad 3 \leq k \leq n! - 3,$$

as $n \rightarrow \infty$. Therefore, away from the exceptional cases $k = 0, 1, 2, n! - 2, n! - 1$, and $n!$, most orbits have size asymptotic to the maximal possible size $(n!)^2$. Furthermore, the authors show that neither the range for k nor the rate of convergence can be improved uniformly over that range.

The proof strategy rests on Burnside's lemma and a detailed analysis of fixed subsets. The authors first establish a general result for a finite group G acting by automorphisms on a finite set X . If

$$T_g(j) = \{x \in X : g^j x = x, g^i x \neq x \text{ for } 1 \leq i < j\},$$

then the number of k -element subsets of X fixed by g is

$$\sum_{\lambda \vdash k} \prod_{j=1}^k \binom{|T_g(j)|/j}{m_j(\lambda)},$$

where $m_j(\lambda)$ is the multiplicity of j in the partition λ . This generalizes a formula of P. W. Diaconis, J. Fulman and R. M. Guralnick [*J. Algebraic Combin.* **28** (2008), no. 1, 189–218; MR2420785] for the action of S_n on $[n]$. Applying this formula to the two-sided action and grouping permutations according to cycle type yields an explicit expression for $T(n, k)$.

The main asymptotic also yields a Gaussian scaling limit near $k \approx \frac{n!}{2}$. This follows by

combining the main theorem with the de Moivre–Laplace approximation for binomial coefficients. The authors also obtain an asymptotic log-concavity statement: for all real x and y , the ratio

$$\frac{T(n, k_n(x))^2}{(T(n, k_n(x+y))T(n, k_n(x-y)))}$$

converges to $e^{y^2} \geq 1$.

The paper closes with six open problems, including the distribution of the number of fixed k -element subsets for a random pair of permutations, eventual log-concavity, the existence of a threshold N beyond which B_n is log-concave, and possible higher-order asymptotic expansions of $T(n, k)$.

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