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Arithmetic complexity of Erdős matrices

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ABSTRACT

Erdős matrices are the doubly stochastic matrices for which equality holds in the Marcus–Ree inequality $\operatorname{maxtrace}(M) \geq \|M\|_{\mathrm{F}}^2$. Every such matrix has rational entries, but little is known about the growth of the denominators that occur. We initiate the study of this arithmetic complexity. Using a block-diagonal construction involving the matrices T_{p+1} (zero diagonal, constant off-diagonal entries $1/p$), where p ranges over the primes, we prove that the largest denominator D_n of an $n \times n$ Erdős matrix satisfies $D_n \geq \exp\left(c\sqrt{n \log n}\right)$ for some constant $c > 0$ and all $n \geq 1$. On the other hand, by relating an Erdős matrix to the Laplacian of its bipartite support graph and applying the Matrix–Tree Theorem, we establish the upper bound $D_n \leq n^{2n} = \exp(2n \log n)$. As a corollary, we obtain that the number of distinct denominators up to dimension N is at least $c\sqrt{N/\log N}$. We also observe that the Gram matrix associated with a Birkhoff decomposition admits a natural representation-theoretic decomposition into trivial and standard components.

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1. Introduction

A classical theorem of Birkhoff [7] states that every doubly stochastic matrix can be written as a convex combination of permutation matrices, that is,

$$A = \sum_{i=1}^{m} x_i P_i, \qquad x_i > 0, \qquad \sum_{i=1}^{m} x_i = 1,$$

where $P_1, \ldots, P_m$ are permutation matrices. We shall refer to any such representation as a *Birkhoff decomposition* of A. Equivalently, the set of all $n \times n$ doubly stochastic matrices forms a convex polytope, known as the *Birkhoff polytope* and denoted Ω_n, whose vertices are precisely the permutation matrices. This polytope continues to attract considerable attention in contemporary matrix theory; see, for example, [2,4,5,9–15,20].

In 1959, Marcus and Ree [19] established the inequality

$$\operatorname{maxtrace}(D) \;\geq\; \|D\|_{\mathrm{F}}^2$$

for every doubly stochastic matrix D, where

$$\operatorname{maxtrace}(D) := \max_{\sigma \in S_n} \sum_{i=1}^{n} d_{i,\sigma(i)}$$

denotes the largest diagonal sum obtained by selecting exactly one entry from each row and each column of D, the maximum being taken over the symmetric group S_n of all permutations of $\{1, \ldots, n\}$, and

$$\|D\|_{\mathrm{F}} := \left(\sum_{i,j=1}^{n} |d_{ij}|^2 \right)^{1/2}$$

denotes the Frobenius norm. Erdős subsequently asked for a characterization of the matrices for which equality holds. Following the terminology introduced by Tripathi [21], such matrices are now called *Erdős matrices*.

Although the Marcus–Ree inequality is more than sixty years old, the systematic study of Erdős matrices has emerged only recently. A modern investigation began with the complete classification of the 3×3 case in [3]. Subsequent work by Tripathi [21], Kushwaha and Tripathi [17], and Karmakar, Krishna, Pal and Krishna Teja [16] revealed a remarkably rigid structure underlying these matrices.

Among the most striking recent discoveries is the fact that an Erdős matrix is completely determined by the positions of its zero entries. Equivalently, every face of the Birkhoff polytope contains at most one Erdős matrix. This rigidity phenomenon transforms the classification problem into a combinatorial problem concerning admissible support patterns and suggests the presence of a much deeper underlying structure. Recent work has also established that every Erdős matrix has rational entries and has provided effective algorithms for their enumeration in small dimensions [16,21].

These developments naturally raise arithmetic questions. Since every Erdős matrix is rational, one may ask how complicated its entries can be from an arithmetic point of view. Karmakar, Krishna, Pal and Krishna Teja [16] observed that surprisingly large denominators already appear in small dimensions and explicitly asked how the corresponding denominator sequence grows. More precisely, if

$$D_n := \max\left\{ \operatorname{lcd}(E) : E \text{ is an } n \times n \text{ Erdős matrix} \right\},$$

where $\operatorname{lcd}(E)$ denotes the least common denominator of the nonzero entries of E, then little appears to be known about the asymptotic behavior of D_n.

The purpose of the present paper is to initiate the study of this arithmetic complexity. We establish the first general lower and upper bounds for the sequence $(D_n)_{n\geq 1}$. On the one hand, we construct Erdős matrices whose denominators grow superpolynomially, obtaining

$$D_n \geq \exp\left(c\sqrt{n\log n}\right)$$

for all $n \geq 1$. On the other hand, we show that

$$D_n \leq n^{2n} = \exp(2n\log n),$$

thereby placing the growth of D_n between two explicit asymptotic scales.

The proof of the upper bound reveals an unexpected connection between Erdős matrices and algebraic graph theory. Using the Brualdi–Dahl characterization of RCDS matrices, the arithmetic of an Erdős matrix can be related to the Laplacian of its bipartite support graph. The matrix-tree theorem then yields a bound in terms of spanning-tree counts.

The paper is organized as follows. Section 2 establishes the notation and collects the preliminary definitions and results needed in the sequel. Section 3 is devoted to the arithmetic complexity function D_n and contains the proof of the main theorem. Section 4 studies the distribution of denominators arising from Erdős matrices. Section 5 presents a representation-theoretic viewpoint based on Gram matrices of permutations and Smith normal forms. The final section contains several open problems.

2. Definitions and preliminary results

Recall that a square matrix is said to be *doubly stochastic* if its entries are nonnegative and all its row and column sums are equal to 1.

Following Tripathi [21], a doubly stochastic matrix E is called an *Erdős matrix* if equality holds in the Marcus–Ree inequality

$$\operatorname{maxtrace}(E) \;=\; \|E\|_{\mathrm{F}}^2.$$

Since every Erdős matrix is rational [16], it is natural to investigate the arithmetic complexity of its entries.

Definition 2.1. Let E be an Erdős matrix with rational entries. The *least common denominator* of E, denoted by

$$\operatorname{lcd}(E),$$

is the smallest positive integer m such that every entry of mE is an integer.

The main object of study of the present paper is the sequence

$$D_n \;=\; \max\Big\{ \operatorname{lcd}(E) : \; E \text{ is an } n \times n \text{ Erdős matrix} \Big\}.$$

The existence of D_n follows from the fact that there are only finitely many Erdős matrices of a given order [16].

We shall repeatedly use the fact that the class of Erdős matrices is stable under block-diagonal sums.

Proposition 2.2. *Let A and B be Erdős matrices. Then*

$$A \oplus B$$

is again an Erdős matrix.

Proof. Since A and B are Erdős matrices,

$$\operatorname{maxtrace}(A) = \|A\|_{\mathrm{F}}^2, \qquad \text{and} \qquad \operatorname{maxtrace}(B) = \|B\|_{\mathrm{F}}^2.$$

For block-diagonal matrices,

$$\operatorname{maxtrace}(A \oplus B) \;=\; \operatorname{maxtrace}(A) + \operatorname{maxtrace}(B).$$

Indeed, the entries of $A \oplus B$ lying outside the two diagonal blocks are zero, so any permutation selects, within each diagonal block, a partial diagonal of that block (a set of entries no two of which lie in the same row or column). Since the entries are nonnegative, a partial diagonal of a square matrix can be completed to a full diagonal without decreasing its sum, so the contributions of the two blocks are at most $\operatorname{maxtrace}(A)$ and $\operatorname{maxtrace}(B)$, respectively. Conversely, combining a maximizing permutation of the first block with a maximizing permutation of the second block yields the reverse inequality.

Similarly,

$$\|A \oplus B\|_{\mathrm{F}}^2 \;=\; \|A\|_{\mathrm{F}}^2 + \|B\|_{\mathrm{F}}^2.$$

Therefore

$$\begin{aligned}
\operatorname{maxtrace}(A \oplus B) \;&=\; \operatorname{maxtrace}(A) + \operatorname{maxtrace}(B) \\
&=\; \|A\|_{\mathrm{F}}^2 + \|B\|_{\mathrm{F}}^2 \\
&=\; \|A \oplus B\|_{\mathrm{F}}^2.
\end{aligned}$$

Hence $A \oplus B$ is an Erdős matrix. □

The following proposition will also be useful.

Proposition 2.3. *Let A and B be rational matrices. Then*

$$\operatorname{lcd}(A \oplus B) \;=\; \operatorname{lcm}\big(\operatorname{lcd}(A), \operatorname{lcd}(B)\big).$$

Proof. Let

$$\alpha \;=\; \operatorname{lcd}(A), \qquad \text{and} \qquad \beta = \operatorname{lcd}(B).$$

Then $k(A \oplus B)$ has integer entries if and only if both kA and kB have integer entries. Hence k must be a common multiple of α and β. The smallest such integer is

$$\operatorname{lcm}(\alpha, \beta),$$

which proves the result. □

Finally, we recall a standard consequence of the Prime Number Theorem.

Proposition 2.4. *Let*

$$p_1 \;<\; p_2 \;<\; \cdots \;<\; p_k$$

denote the first k prime numbers.

Then the sum of the first k primes grows on the order of $k^2 \log k$, while the sum of their logarithms grows on the order of $k \log k$. More precisely, there exist positive constants c_1, c_2, c_3, c_4 such that

$$c_1 k^2 \log k \;\le\; \sum_{i=1}^{k} p_i \;\le\; c_2 k^2 \log k$$

and

$$c_3 k \log k \;\le\; \sum_{i=1}^{k} \log p_i \;\le\; c_4 k \log k$$

for all sufficiently large k.

Proof. This is a standard consequence of the Prime Number Theorem; see, for instance, [1, Chapter 1]. Indeed, the Prime Number Theorem implies $p_i \sim i \log i$, and the two estimates follow by summing p_i and $\log p_i$, respectively. □

3. Arithmetic complexity of Erdős matrices

We are now ready to investigate the growth of the arithmetic complexity function

$$D_n \;=\; \max\Big\{ \operatorname{lcd}(E) : \; E \text{ is an } n \times n \text{ Erdős matrix} \Big\}.$$

Our main result is the following.

Theorem 3.1. *There exists a constant $c > 0$ such that*

$$D_n \;\ge\; \exp\left(c\sqrt{n \log n}\right)$$

for every $n \ge 1$. In particular the sequence (D_n) grows faster than every polynomial.

Moreover,

$$D_n \;\le\; n^{2n} \;=\; \exp(2n \log n)$$

for every $n \ge 1$.

The theorem provides both lower and upper bounds for the arithmetic complexity sequence $(D_n)_{n\ge1}$. The lower bound is obtained by combining the block-diagonal construction of Proposition 2.2 with an explicit family of Erdős matrices having prescribed denominators. The upper bound relies on the rigidity theory developed in [16] together with a graph-theoretic interpretation of the Brualdi–Dahl representation of RCDS matrices [8].

The proof is divided accordingly. We first establish the lower bound by constructing Erdős matrices with large least common denominators. We then prove the upper bound by relating Erdős matrices to Laplacians of bipartite support graphs and applying the matrix-tree theorem.

3.1. Proof of the superpolynomial lower bound

We begin by establishing the lower bound in Theorem 3.1. The argument is organized in three steps. First, we construct Erdős matrices with large least common denominators along an infinite subsequence of dimensions. Second, we fill the gaps between these dimensions by adding identity blocks. Finally, we absorb the finitely many small values of n by decreasing the constant.

Step 1: The lower bound along a subsequence.

Step 1a: Construction of a basic Erdős matrix.

For every integer $m \geq 2$, define

$$T_m := \frac{1}{m-1}\big(mJ_m - I_m\big),$$

where J_m denotes the $m \times m$ matrix whose entries are all equal to $\frac{1}{m}$. Equivalently,

$$(T_m)_{ij} = \begin{cases} 0, & i = j, \\ \frac{1}{m-1}, & i \neq j. \end{cases}$$

Each row and column of T_m contains exactly $m-1$ nonzero entries, each equal to $\frac{1}{(m-1)}$, and therefore T_m is doubly stochastic. Moreover,

$$\|T_m\|_{\mathrm{F}}^2 = m(m-1)\left(\frac{1}{m-1}\right)^2 = \frac{m}{m-1}.$$

Since $m \geq 2$, there exists a derangement $\sigma \in S_m$. For such a permutation,

$$\sum_{i=1}^{m}(T_m)_{i,\sigma(i)} = m \cdot \frac{1}{m-1} = \frac{m}{m-1}.$$

On the other hand, every entry of T_m is at most $\frac{1}{(m-1)}$, so every permutation sum is bounded above by $\frac{m}{(m-1)}$. Hence

$$\mathrm{maxtrace}(T_m) = \frac{m}{m-1} = \|T_m\|_{\mathrm{F}}^2.$$

Thus T_m is an Erdős matrix. Furthermore,

$$\mathrm{lcd}(T_m) = m-1.$$

Step 1b: Block-diagonal construction using primes.

Let

$$p_1 < p_2 < \cdots < p_k$$

denote the first k prime numbers, and define

$$A_k := T_{p_1+1} \oplus T_{p_2+1} \oplus \cdots \oplus T_{p_k+1}.$$

By Proposition 2.2, the matrix A_k is an Erdős matrix. Its dimension is

$$n_k = \sum_{i=1}^{k}(p_i + 1) = k + \sum_{i=1}^{k} p_i.$$

Since

$$\operatorname{lcd}(T_{p_i+1}) = p_i$$

for each i, and since the numbers $p_1, \dots, p_k$ are distinct primes, we obtain

$$\operatorname{lcd}(A_k) = p_1 p_2 \cdots p_k.$$

Consequently,

$$\log \operatorname{lcd}(A_k) = \sum_{i=1}^{k} \log p_i.$$

Step 1c: Asymptotic estimates.

By Proposition 2.4, there exist positive constants c_1, c_2, c_3, c_4 such that

$$c_1 k^2 \log k \leq \sum_{i=1}^{k} p_i \leq c_2 k^2 \log k$$

and

$$c_3 k \log k \leq \sum_{i=1}^{k} \log p_i \leq c_4 k \log k$$

for all sufficiently large k.

Since

$$n_k = k + \sum_{i=1}^{k} p_i,$$

it follows that there exist positive constants c_5 and c_6 such that

$$c_5 k \log k \leq \sqrt{n_k \log n_k} \leq c_6 k \log k$$

for all sufficiently large k.

Combining these estimates, we obtain

$$\log \operatorname{lcd}(A_k) \;\geq\; c\sqrt{n_k \log n_k}$$

for some positive constant c.

Exponentiating gives

$$\operatorname{lcd}(A_k) \;\geq\; \exp\left(c\sqrt{n_k \log n_k}\right).$$

Step 1d: Conclusion along the subsequence.

Since A_k is an $n_k \times n_k$ Erdős matrix,

$$D_{n_k} \;\geq\; \operatorname{lcd}(A_k).$$

Therefore,

$$D_{n_k} \;\geq\; \exp\left(c\sqrt{n_k \log n_k}\right)$$

for all sufficiently large k.

Step 2: Extension to all sufficiently large dimensions.

We now extend the preceding estimate from the subsequence (n_k) to all sufficiently large integers n.

Let n be large, and choose k such that

$$n_k \;\leq\; n \;<\; n_{k+1}.$$

Define

$$B_n := \; A_k \oplus I_{n-n_k}.$$

The identity matrix I_{n-n_k} is an Erdős matrix, and by Proposition 2.2, B_n is also an Erdős matrix. Moreover,

$$\operatorname{lcd}(I_{n-n_k}) \;=\; 1,$$

and therefore Proposition 2.3 gives

$$\operatorname{lcd}(B_n) \;=\; \operatorname{lcm}\left(\operatorname{lcd}(A_k), 1\right) \;=\; \operatorname{lcd}(A_k).$$

Hence

$$D_n \;\geq\; \operatorname{lcd}(B_n) \;=\; \operatorname{lcd}(A_k).$$

It remains to compare n and n_k. Since

$$n_{k+1} - n_k \;=\; p_{k+1} + 1,$$

we have

$$0 \;\leq\; n - n_k \;<\; p_{k+1} + 1.$$

By the Prime Number Theorem, there exists a constant $C > 0$ such that

$$p_{k+1} + 1 \;\leq\; Ck \log k$$

for all sufficiently large k.

On the other hand, by Proposition 2.4, there exists a constant $c > 0$ such that

$$n_k \;\geq\; c\, k^2 \log k$$

for all sufficiently large k.

Therefore

$$n \;<\; n_k + Ck \log k \;\leq\; n_k + \frac{C}{c}\,\frac{n_k}{k}.$$

Since $k \geq 1$, it follows that

$$n \;\leq\; \left(1 + \frac{C}{c}\right) n_k$$

for all sufficiently large k.

Hence there exists a constant $M > 1$ such that

$$n \;\leq\; M n_k$$

for all sufficiently large k.

Since the function

$$x \mapsto \sqrt{x \log x}$$

is increasing for $x \geq 2$, we obtain

$$\sqrt{n \log n} \;\leq\; \sqrt{M n_k \log(M n_k)}.$$

Since M is fixed, we have

$$\log(M n_k) \;=\; \log M + \log n_k \;\leq\; 2 \log n_k$$

for all sufficiently large k. Hence

$$\sqrt{n\log n} \;\le\; \sqrt{2M}\,\sqrt{n_k\log n_k}.$$

Consequently,

$$\sqrt{n_k\log n_k} \;\ge\; \frac{1}{\sqrt{2M}}\sqrt{n\log n}.$$

Combining this estimate with Step 1c, we obtain

$$\log\big(\operatorname{lcd}(A_k)\big) \;\ge\; \frac{c}{\sqrt{2M}}\sqrt{n\log n}.$$

After decreasing the constant $c > 0$, if necessary, this gives

$$\log\big(\operatorname{lcd}(A_k)\big) \;\ge\; c\sqrt{n\log n}.$$

Thus

$$D_n \;\ge\; \operatorname{lcd}(A_k) \;\ge\; \exp\big(c\sqrt{n\log n}\big)$$

for all sufficiently large n.

Step 3: Extension to all $n \ge 1$.

The preceding step proves that there exist constants $c > 0$ and $N \ge 2$ such that

$$D_n \;\ge\; \exp\big(c\sqrt{n\log n}\big)$$

for every $n \ge N$. It remains only to handle the finitely many values $1 \le n < N$. Observe that $D_n > 1$ for every $n \ge 2$. Indeed, every diagonal of the matrix J_n introduced in Step 1a has sum $n \cdot \frac{1}{n} = 1$, so that $\operatorname{maxtrace}(J_n) = 1 = \|J_n\|_{\mathrm{F}}^2$; hence J_n is an $n \times n$ Erdős matrix with $\operatorname{lcd}(J_n) = n$, and $D_n \ge n$. We may therefore decrease c, if necessary, so that

$$D_n \;\ge\; \exp\big(c\sqrt{n\log n}\big)$$

also holds for $2 \le n < N$. Finally, for $n = 1$, we have $\sqrt{n\log n} = 0$ and $D_1 = 1$, so the same inequality holds. Therefore, after possibly replacing c by a smaller positive constant, we obtain

$$D_n \;\ge\; \exp\big(c\sqrt{n\log n}\big)$$

for every $n \ge 1$.

This proves the first assertion of Theorem 3.1.

3.2. Proof of the upper bound

We now prove the second assertion of Theorem 3.1, namely

$$D_n \leq n^{2n} \;=\; \exp(2n \log n).$$

Let E be an $n \times n$ Erdős matrix. We first consider the case where E is indecomposable, that is, where there exist no permutation matrices P and Q such that PEQ is a nontrivial block-diagonal matrix (the rows and the columns being permuted independently). The general case will be deduced later by applying the resulting estimate to the indecomposable blocks in the canonical block decomposition of E.

Step 1: The indecomposable case.

Step 1a: The Brualdi–Dahl characterization.

Let

$$S := \; \mathrm{Skel}(E)$$

denote the skeleton of E, that is, the binary matrix defined by

$$S_{ij} \;=\; \begin{cases} 1, & E_{ij} > 0, \\ 0, & E_{ij} = 0. \end{cases}$$

Since E is an Erdős matrix, Proposition 2.7 of [16] implies that E has *restricted common diagonal sums.* Recall that a square nonnegative matrix is said to have restricted common diagonal sums (RCDS) if the sum of the entries selected by any permutation avoiding the zero entries of the matrix is independent of the permutation.

A result of Brualdi and Dahl [8, Theorem 2.1] (see also [16, Theorem 1.4]) implies, in the present setting, that there exist vectors

$$u := \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \qquad \text{and} \qquad v := \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix},$$

such that

$$E_{ij} \;=\; (u_i + v_j) S_{ij}$$

for all i, j.

Thus every nonzero entry of E is determined by the sum of a row parameter u_i and a column parameter v_j, while the zero pattern of E is completely encoded by its skeleton S.

To encode the zero pattern of E, we associate with the skeleton S a bipartite graph Γ_S. The vertices of Γ_S consist of two disjoint sets:

$$\{r_1, \ldots, r_n\} \qquad \text{and} \qquad \{c_1, \ldots, c_n\},$$

where r_i represents the i-th row of S and c_j represents the j-th column of S.

A row vertex r_i is joined by an edge to a column vertex c_j whenever the corresponding entry of S is nonzero, that is,

$$r_i \sim c_j \qquad \Longleftrightarrow \qquad S_{ij} = 1.$$

Thus the edges of Γ_S record precisely the positions of the nonzero entries of E (or equivalently of S).

Since E is indecomposable, its support graph Γ_S must be connected. Indeed, suppose that Γ_S were disconnected, and let one of its connected components have row-vertex set R and column-vertex set C (every vertex of Γ_S has positive degree, since E has no zero row or column). Every nonzero entry of E lying in a row of R lies in a column of C, and conversely. Summing the entries of the corresponding submatrix first by rows and then by columns gives, since E is doubly stochastic,

$$|R| \;=\; \sum_{i\in R}\sum_{j\in C} E_{ij} \;=\; |C|.$$

Permuting the rows and the columns of E so that the row and column sets of each connected component become consecutive therefore transforms E into a nontrivial block-diagonal matrix with square diagonal blocks, contradicting the indecomposability of E. Hence Γ_S is connected.

Step 1b: A Laplacian formulation.

The representation obtained in Step 1a expresses every nonzero entry of E in terms of the parameters

$$u_1, \ldots, u_n, \qquad \text{and} \qquad v_1, \ldots, v_n.$$

We now show that these parameters satisfy a linear system associated with the support graph Γ_S.

Let

$$d_i := \deg(r_i), \qquad \text{and} \qquad e_j := \deg(c_j)$$

denote the degrees of the row and column vertices of Γ_S, respectively. Equivalently,

$$d_i = \sum_{j=1}^{n} S_{ij}, \qquad \text{and} \qquad e_j = \sum_{i=1}^{n} S_{ij}.$$

Since E is doubly stochastic, every row sum and every column sum is equal to 1. Consider first the i-th row. Using the representation

$$E_{ij} \;=\; (u_i + v_j) S_{ij},$$

we obtain

$$1 \;=\; \sum_{j=1}^{n} E_{ij} \;=\; \sum_{\substack{j \\ \text{such that} \\ S_{ij}=1}} (u_i + v_j).$$

Since exactly d_i terms contribute to the sum, this becomes

$$d_i u_i + \sum_{\substack{j \\ \text{such that} \\ S_{ij}=1}} v_j \;=\; 1, \qquad 1 \le i \le n. \tag{3.1}$$

Similarly, for the j-th column,

$$1 \;=\; \sum_{i=1}^{n} E_{ij} \;=\; \sum_{\substack{i \\ \text{such that} \\ S_{ij}=1}} (u_i + v_j),$$

which yields

$$\sum_{\substack{i \\ \text{such that} \\ S_{ij}=1}} u_i + e_j v_j \;=\; 1, \qquad 1 \le j \le n. \tag{3.2}$$

Collecting these $2n$ equations gives a linear system for the $2n$ unknowns

$$u_1, \ldots, u_n, v_1, \ldots, v_n.$$

Recall that

$$u := \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \qquad \text{and} \qquad v := \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix},$$

and define

$$\Delta_r := \operatorname{diag}(d_1, \ldots, d_n), \qquad \text{and} \qquad \Delta_c := \operatorname{diag}(e_1, \ldots, e_n).$$

Observe that the i-th coordinate of the vector Sv is

$$(Sv)_i \;=\; \sum_{j=1}^{n} S_{ij} v_j \;=\; \sum_{\substack{j \\ \text{such that} \\ S_{ij}=1}} v_j,$$

while the j-th coordinate of $S^\mathsf{T} u$ is

$$(S^\mathsf{T} u)_j \;=\; \sum_{i=1}^{n} S_{ij} u_i \;=\; \sum_{\substack{i \\ \text{such that} \\ S_{ij}=1}} u_i.$$

Moreover, the vectors $\Delta_r u$ and $\Delta_c v$ have coordinates $d_i u_i$ and $e_j v_j$, respectively.

Combining the n equations in (3.1) with the n equations in (3.2), we obtain the block linear system

$$H \begin{pmatrix} u \\ v \end{pmatrix} \;=\; \mathbf{1},$$

where

$$H \;=\; \begin{pmatrix} \Delta_r & S \\ S^\mathsf{T} & \Delta_c \end{pmatrix}.$$

This matrix has a natural graph-theoretic interpretation. Indeed, H is precisely the signless Laplacian of the bipartite graph Γ_S (see, for example, [6, Chapter 7]).

Since Γ_S is bipartite, changing the signs of the coordinates corresponding to the column vertices transforms the signless Laplacian into the ordinary graph Laplacian. More precisely, if

$$P := \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix},$$

then

$$PHP \;=\; \begin{pmatrix} \Delta_r & -S \\ -S^\mathsf{T} & \Delta_c \end{pmatrix} =: L.$$

The matrix L is precisely the ordinary graph Laplacian of Γ_S.

Step 1c: Normalization of the parameters.

The linear system obtained in Step 1b is not uniquely solvable. Indeed, the representation

$$E_{ij} \;=\; (u_i + v_j) S_{ij}$$

remains unchanged if we replace

$$u_i \mapsto u_i + t, \qquad \text{and} \qquad v_j \mapsto v_j - t,$$

for an arbitrary real number t. Since

$$(u_i + t) + (v_j - t) \ = \ u_i + v_j,$$

the resulting parameters determine exactly the same matrix E.

To relate the system obtained above to the graph Laplacian, define

$$y := \ P\begin{pmatrix} u \\ v \end{pmatrix} \ = \ \begin{pmatrix} u \\ -v \end{pmatrix}.$$

Since

$$H\begin{pmatrix} u \\ v \end{pmatrix} \ = \ \mathbf{1}$$

and

$$L \ = \ PHP,$$

it follows that

$$Ly \ = \ P\mathbf{1}.$$

The vector $P\mathbf{1}$ has integer entries, namely 1 on the row vertices and -1 on the column vertices.

To obtain a unique representative of the solution, we impose the normalization

$$v_n = 0.$$

Equivalently,

$$y_{c_n} = 0.$$

Since Γ_S is connected, the kernel of its Laplacian L is one-dimensional and spanned by the all-one vector (see, for example, [6, Proposition 1.3.1]). Hence one of the equations in the system $Ly = P\mathbf{1}$ is redundant.

Removing the variable corresponding to c_n and deleting the equation corresponding to c_n, we obtain a square linear system

$$L^{(n)}z \ = \ b,$$

where $L^{(n)}$ is obtained from L by deleting the row and column corresponding to c_n, and b is an integer vector. In particular, $L^{(n)}$ is a principal cofactor matrix of L.

Step 1d: Application of the matrix-tree theorem.

By the Matrix–Tree Theorem (see, for example, [6, Section 1.3.5]), the determinant of every principal cofactor matrix of the Laplacian of a connected graph is equal to the number of spanning trees of the graph. Hence

$$\det\left(L^{(n)}\right) \;=\; \tau(\Gamma_S),$$

where $\tau(\Gamma_S)$ denotes the number of spanning trees of Γ_S.

Since Γ_S is connected, it possesses at least one spanning tree. Therefore

$$\tau(\Gamma_S) \;\geq\; 1,$$

and consequently

$$\det\left(L^{(n)}\right) \;\neq\; 0.$$

Thus the reduced system is nonsingular.

By Cramer's rule, every coordinate of z can be written as

$$\frac{\det(M)}{\det\left(L^{(n)}\right)},$$

where M is an integer matrix obtained by replacing one column of $L^{(n)}$ with the right-hand side vector of the reduced system. Since

$$\det\left(L^{(n)}\right) \;=\; \tau(\Gamma_S),$$

it follows that every coordinate of z belongs to

$$\frac{1}{\tau(\Gamma_S)}\mathbb{Z}.$$

Since z consists precisely of the parameters u_i and $-v_j$ $(j \neq n)$, and since $v_n = 0$, the same conclusion holds for all parameters u_i and v_j.

Consequently,

$$u_i,\; v_j \in \frac{1}{\tau(\Gamma_S)}\mathbb{Z}.$$

Since $S_{ij} \in \{0, 1\}$, it follows that

$$E_{ij} \;=\; (u_i + v_j)S_{ij} \in \frac{1}{\tau(\Gamma_S)}\mathbb{Z}.$$

Thus multiplying E by $\tau(\Gamma_S)$ clears the denominators of all entries, and hence

$$\operatorname{lcd}(E) \;\le\; \tau(\Gamma_S).$$

Step 1e: Bounding the number of spanning trees.

Since Γ_S is a spanning subgraph of $K_{n,n}$, every spanning tree of Γ_S is naturally a spanning tree of $K_{n,n}$. Hence

$$\tau(\Gamma_S) \;\le\; \tau(K_{n,n}).$$

The classical formula of Lewis [18] for the number of spanning trees of the complete bipartite graph (see, for example, [22, Section 4.2]) gives

$$\tau(K_{n,n}) \;=\; n^{2n-2}.$$

Thus every indecomposable $n \times n$ Erdős matrix satisfies

$$\operatorname{lcd}(E) \;\le\; n^{2n-2}.$$

Step 2: The general case.

Now let E be an arbitrary $n \times n$ Erdős matrix.

As in Step 1a, the connected components of the support graph of E determine permutation matrices P and Q such that

$$PEQ \;=\; E_1 \oplus E_2 \oplus \cdots \oplus E_r,$$

where the diagonal blocks $E_1, \ldots, E_r$ are square, doubly stochastic and indecomposable, of sizes

$$n_1, \ldots, n_r, \qquad \text{with} \qquad n_1 + \cdots + n_r = n.$$

The diagonals of PEQ are obtained from the diagonals of E by relabeling, so that $\operatorname{maxtrace}(PEQ) = \operatorname{maxtrace}(E)$; the Frobenius norm and the least common denominator are likewise invariant under permutations of the rows and of the columns. Hence $E_1 \oplus \cdots \oplus E_r$ is again an Erdős matrix, and $\operatorname{lcd}(E) = \operatorname{lcd}(E_1 \oplus \cdots \oplus E_r)$.

Since

$$\operatorname{maxtrace}(E_1 \oplus \cdots \oplus E_r) \;=\; \sum_{k=1}^{r} \operatorname{maxtrace}(E_k)$$

by the argument in the proof of Proposition 2.2, and

$$\|E_1 \oplus \cdots \oplus E_r\|_{\mathrm{F}}^2 \;=\; \sum_{k=1}^{r} \|E_k\|_{\mathrm{F}}^2,$$

while $\operatorname{maxtrace}(E_k) \geq \|E_k\|_{\mathrm{F}}^2$ for every k by the Marcus–Ree inequality, equality for $E_1 \oplus \cdots \oplus E_r$ forces equality on each block. Hence each E_k is itself an Erdős matrix.

Applying the estimate of Step 1 to each block, and using Proposition 2.3 (applied inductively) together with the fact that the least common multiple of finitely many positive integers divides their product, we obtain

$$\operatorname{lcd}(E) \;=\; \operatorname{lcm}\big(\operatorname{lcd}(E_1), \ldots, \operatorname{lcd}(E_r)\big) \;\leq\; \prod_{k=1}^{r} \operatorname{lcd}(E_k) \;\leq\; \prod_{k=1}^{r} n_k^{2n_k-2}.$$

Since $n_k \leq n$ for every k, we obtain

$$\prod_{k=1}^{r} n_k^{2n_k-2} \;\leq\; \prod_{k=1}^{r} n^{2n_k} \;=\; n^{2n}.$$

Hence

$$\operatorname{lcd}(E) \;\leq\; n^{2n}$$

for every $n \times n$ Erdős matrix E.

Taking the maximum over all such matrices gives

$$D_n \;\leq\; n^{2n}.$$

This proves the second assertion of Theorem 3.1.

4. Distinct denominators

The arithmetic complexity function

$$D_n \;=\; \max\Big\{\operatorname{lcd}(A) : A \text{ is an } n \times n \text{ Erdős matrix}\Big\}$$

measures the largest denominator that can occur in a given dimension. A complementary question is to understand how many different denominators occur altogether.

To this end, define

$$\mathcal{D}(N) := \;\#\Big\{\operatorname{lcd}(A) : A \text{ is an } n \times n \text{ Erdős matrix with } n \leq N\Big\}.$$

The construction used in the proof of Theorem 3.1 provides a family of Erdős matrices whose denominators are pairwise distinct. This yields the following estimate.

Corollary 4.1. *There exists a constant $c > 0$ such that*

$$\mathcal{D}(N) \;\geq\; c\sqrt{\frac{N}{\log N}}$$

for all sufficiently large N.

Proof. In the proof of Theorem 3.1, we constructed a family of Erdős matrices $\{A_k\}_{k\geq 1}$ satisfying

$$A_k \in \mathbb{R}^{n_k \times n_k}, \qquad \operatorname{lcd}(A_k) = p_1 p_2 \cdots p_k,$$

where $p_1, \ldots, p_k$ are the first k primes and

$$n_k \;=\; \sum_{i=1}^{k} (p_i + 1).$$

Since $p_1, \ldots, p_k$ are distinct primes, the integers

$$\operatorname{lcd}(A_k) \;=\; p_1 p_2 \cdots p_k$$

are pairwise distinct.

Furthermore,

$$n_k \;=\; k + \sum_{i=1}^{k} p_i.$$

By Proposition 2.4, there exists a constant $C > 0$ such that

$$n_k \;\leq\; Ck^2 \log k$$

for all sufficiently large k.

Consequently, every integer k satisfying

$$Ck^2 \log k \;\leq\; N$$

also satisfies $n_k \leq N$. It follows that the number of indices k for which $n_k \leq N$ is bounded below by a constant multiple of

$$\sqrt{\frac{N}{\log N}}$$

for all sufficiently large N.

Each such index k contributes a distinct denominator arising from an Erdős matrix of order at most N. Therefore

$$\mathcal{D}(N) \;\geq\; c\sqrt{\frac{N}{\log N}}$$

for all sufficiently large N. $\square$

5. Representation theory and arithmetic complexity

In this section we describe a second approach to the arithmetic complexity of Erdős matrices.

Every doubly stochastic matrix admits a representation as a convex combination of permutation matrices. The corresponding Gram matrix of the supporting permutations contains surprisingly rich algebraic information. We show that this Gram matrix naturally decomposes according to the representation theory of the symmetric group and that its Smith normal form imposes arithmetic constraints on the denominators of Erdős matrices.

We first collect some standard notation from the representation theory of the symmetric group S_n. For $\sigma \in S_n$, we write

$$\mathrm{Fix}(\sigma) := \big\{ i \in \{1, \ldots, n\} : \; \sigma(i) = i \big\}$$

for the set of fixed points of σ, and $\#\,\mathrm{Fix}(\sigma)$ for its cardinality. The natural permutation representation of S_n on $\mathbb{C}^n$ decomposes as the direct sum of the one-dimensional trivial representation and the $(n-1)$-dimensional standard representation. If χ_{perm} denotes the character of the permutation representation and χ_{std} denotes the character of the standard representation, then

$$\chi_{\mathrm{perm}}(\sigma) = \#\,\mathrm{Fix}(\sigma) = 1 + \chi_{\mathrm{std}}(\sigma).$$

Finally, given two complex matrices X and Y of the same size, we write

$$\langle X, Y \rangle_{\mathrm{F}} := \mathrm{tr}(X^* Y)$$

for the Frobenius inner product, so that $\|X\|_{\mathrm{F}}^2 = \langle X, X \rangle_{\mathrm{F}}$.

We shall use these facts in the following proposition.

Proposition 5.1. *Let $\sigma_1, \ldots, \sigma_m$ be permutations in S_n, and let $P_1, \ldots, P_m$ be their corresponding permutation matrices. Define the Gram matrix*

$$M = (M_{ij})_{i,j=1}^m \in \mathbb{Z}^{m \times m}$$

by

$$M_{ij} = \langle P_i, P_j \rangle_{\mathrm{F}} = \#\,\mathrm{Fix}(\sigma_i^{-1}\sigma_j).$$

Then

$$M = J + B,$$

where

$$J = \mathbf{1}\mathbf{1}^{\mathsf{T}}$$

is the $m \times m$ all-ones matrix and

$$B = (B_{ij})_{i,j=1}^{m}, \qquad B_{ij} = \chi_{\mathrm{std}}(\sigma_i^{-1}\sigma_j),$$

with χ_{std} denoting the character of the standard representation of S_n. Moreover,

1. *B is symmetric and positive semidefinite;*
2. *$B_{ij} \in \mathbb{Z}$ for all i, j;*
3. $\mathrm{rank}(B) \leq (n-1)^2$.

Proof. Recall that the permutation character of S_n is given by

$$\chi_{\mathrm{perm}}(\sigma) \;=\; \#\,\mathrm{Fix}(\sigma).$$

Since the permutation representation decomposes as the direct sum of the trivial representation and the standard representation, we have

$$\chi_{\mathrm{perm}} \;=\; \mathbf{1} + \chi_{\mathrm{std}},$$

where $\mathbf{1}$ denotes the trivial character.

Consequently,

$$M_{ij} \;=\; \#\,\mathrm{Fix}(\sigma_i^{-1}\sigma_j) \;=\; 1 + \chi_{\mathrm{std}}(\sigma_i^{-1}\sigma_j),$$

which yields the decomposition

$$M = J + B.$$

To establish the remaining properties, let $\rho_{\mathrm{std}} : S_n \to U(n-1)$ be a unitary representation affording the character χ_{std}. Then

$$\begin{aligned} B_{ij} &= \chi_{\mathrm{std}}(\sigma_i^{-1}\sigma_j) \\ &= \mathrm{tr}\big(\rho_{\mathrm{std}}(\sigma_i^{-1}\sigma_j)\big) \\ &= \mathrm{tr}\big(\rho_{\mathrm{std}}(\sigma_i)^*\rho_{\mathrm{std}}(\sigma_j)\big) \\ &= \big\langle \rho_{\mathrm{std}}(\sigma_i), \rho_{\mathrm{std}}(\sigma_j)\big\rangle_{\mathrm{F}}. \end{aligned}$$

Thus B is the Gram matrix of the matrices

$$\rho_{\mathrm{std}}(\sigma_1), \dots, \rho_{\mathrm{std}}(\sigma_m),$$

and is therefore Hermitian and positive semidefinite.

Since

$$\chi_{\mathrm{std}}(\sigma) \;=\; \#\,\mathrm{Fix}(\sigma) - 1,$$

the entries of B are integers; in particular, B is real symmetric.

Finally, the matrices $\rho_{\mathrm{std}}(\sigma_i)$ belong to the vector space $M_{n-1}(\mathbb{C})$, which has dimension $(n-1)^2$. Since the rank of a Gram matrix cannot exceed the dimension of the ambient space, it follows that

$$\mathrm{rank}(B) \;\le\; (n-1)^2. \quad \square$$

Proposition 5.1 identifies a representation-theoretic component of the Gram matrix associated with a collection of permutations. We now show that the same Gram matrix controls the arithmetic of Erdős matrices through its Smith normal form.

Indeed, the coefficients in a Birkhoff decomposition of an Erdős matrix satisfy a linear system whose coefficient matrix is precisely the Gram matrix M. The invariant factors of M therefore constrain the possible denominators of these coefficients and, consequently, the denominators of the entries of the matrix itself.

Theorem 5.2. *Let A be an $n \times n$ Erdős matrix, and let*

$$A \;=\; \sum_{i=1}^{m} x_i P_i$$

be a Birkhoff decomposition whose supporting permutation matrices $P_1, \dots, P_m$ are linearly independent.

If

$$d_1 \mid d_2 \mid \cdots \mid d_m$$

are the invariant factors of the associated Gram matrix

$$M_{ij} \;=\; \langle P_i, P_j \rangle_{\mathrm{F}},$$

then

$$\operatorname{lcd}(A) \ \mid \ d_m \ \operatorname{lcd}\bigl(\|A\|_{\mathrm{F}}^2\bigr).$$

Proof. Set

$$\lambda \ = \ \|A\|_{\mathrm{F}}^2.$$

Since A has rational entries, we have $\lambda \in \mathbb{Q}$.

Step 1. A linear system for the coefficients.
Because A is an Erdős matrix,

$$\max_{\sigma \in S_n} \langle A, P_\sigma \rangle_{\mathrm{F}} \ = \ \lambda. \tag{5.1}$$

For each i,

$$\langle A, P_i \rangle_{\mathrm{F}} \leq \lambda.$$

On the other hand,

$$\lambda \ = \ \langle A, A \rangle_{\mathrm{F}} \ = \ \sum_{j=1}^{m} x_j \langle A, P_j \rangle_{\mathrm{F}}.$$

Since $x_j > 0$ and $\sum_j x_j = 1$, the right-hand side is a convex combination of numbers bounded above by λ. Hence equality is possible only if

$$\langle A, P_i \rangle_{\mathrm{F}} \ = \ \lambda, \qquad 1 \leq i \leq m.$$

Using

$$A \ = \ \sum_{j=1}^{m} x_j P_j,$$

it follows that

$$\sum_{j=1}^{m} \langle P_i, P_j \rangle_{\mathrm{F}} \, x_j \ = \ \lambda, \qquad 1 \leq i \leq m.$$

Thus

$$M\mathbf{x} \ = \ \lambda \mathbf{1}, \tag{5.2}$$

where

$$\mathbf{x} \;=\; (x_1, \ldots, x_m)^\mathsf{T}.$$

Since the permutation matrices $P_1, \ldots, P_m$ are linearly independent,

$$\mathbf{c}^\mathsf{T} M \mathbf{c} \;=\; \left\| \sum_{i=1}^{m} c_i P_i \right\|_\mathrm{F}^2$$

vanishes only when $\mathbf{c} = 0$. Therefore M is positive definite and hence invertible.

Step 2. Smith normal form.

Let

$$UMV \;=\; D \;=\; \mathrm{diag}(d_1, \ldots, d_m)$$

be the Smith normal form of M, where

$$d_1 \mid d_2 \mid \cdots \mid d_m.$$

Multiplying (5.2) by U and setting

$$\mathbf{y} = V^{-1}\mathbf{x}, \qquad \mathbf{b} = U\mathbf{1},$$

gives

$$D\mathbf{y} \;=\; \lambda \mathbf{b}.$$

Since $\mathbf{b} \in \mathbb{Z}^m$, we obtain

$$d_i y_i \;=\; \lambda b_i, \qquad 1 \le i \le m.$$

Let

$$L \;=\; \mathrm{lcd}(\lambda).$$

Since $d_i \mid d_m$ for every i, each coordinate y_i has denominator dividing Ld_m.

Because V is unimodular, its entries are integers, and therefore the same denominator bound holds for every coordinate of

$$\mathbf{x} \;=\; V\mathbf{y}.$$

Step 3. Denominators of the entries of A.

Finally,

$$A_{rs} \;=\; \sum_{i=1}^{m} x_i (P_i)_{rs},$$

with

$$(P_i)_{rs} \in \{0,1\}.$$

Hence every entry of A is an integer linear combination of the coefficients x_i, and therefore has denominator dividing Ld_m.

It follows that

$$\operatorname{lcd}(A) \mid Ld_m.$$

Since

$$L = \operatorname{lcd}\big(\|A\|_{\mathrm{F}}^2\big),$$

we conclude that

$$\operatorname{lcd}(A) \mid d_m \operatorname{lcd}\big(\|A\|_{\mathrm{F}}^2\big). \quad \square$$

Remark 5.3. Two points regarding the Birkhoff decomposition in Theorem 5.2 deserve clarification.

- *Non-uniqueness and the choice of decomposition.* The Birkhoff decomposition of an Erdős matrix A is generally not unique, and Theorem 5.2 applies to every decomposition

  $$A = \sum_{i=1}^{m} x_i P_i$$

  satisfying its hypotheses. For a fixed admissible decomposition, the resulting divisibility constraint depends on the decomposition through the largest invariant factor d_m of the associated Gram matrix. Since the factor

  $$\operatorname{lcd}\big(\|A\|_{\mathrm{F}}^2\big)$$

  depends only on A, a decomposition with a smaller value of d_m gives a smaller numerical upper bound when Theorem 5.2 is applied to that decomposition alone. The non-uniqueness of the Birkhoff decomposition can, however, provide additional arithmetic information. Indeed, let $\mathcal{D}$ range over the admissible Birkhoff decompositions of A, and let $d(\mathcal{D})$ denote the largest invariant factor of the Gram matrix associated with $\mathcal{D}$. Since Theorem 5.2 holds for every such decomposition,

  $$\operatorname{lcd}(A) \mid d(\mathcal{D}) \operatorname{lcd}\big(\|A\|_{\mathrm{F}}^2\big)$$

  for every $\mathcal{D}$. Consequently,

$$\operatorname{lcd}(A) \mid \operatorname{lcd}\big(\|A\|_{\mathrm{F}}^2\big) \gcd_{\mathcal{D}} d(\mathcal{D}).$$

Thus different admissible decompositions may yield complementary divisibility information. In particular, the search for suitable decompositions can sharpen the conclusion of Theorem 5.2, either by producing a small individual invariant factor or by combining the divisibility constraints arising from several decompositions.

A related, but distinct, consideration is the number m of supporting permutation matrices. Although m does not appear explicitly in the divisibility statement of Theorem 5.2, it enters the subsequent determinant estimates, such as those used in Corollary 5.4. Hence a decomposition with fewer supporting permutation matrices may lead to sharper uniform estimates. Controlling d_m and controlling m are, in general, different optimization problems, and we do not claim that they can always be optimized simultaneously or that there exists a canonical optimal Birkhoff decomposition.

There is nevertheless a canonical geometric object underlying all these choices. The minimal face of the Birkhoff polytope Ω_n containing A is uniquely determined by the support of A. Its vertices are precisely the permutation matrices whose supports are contained in the support of A. Every Birkhoff decomposition of A with positive coefficients therefore uses only vertices from this fixed minimal face. Thus the relevant choice is not among different faces of the Birkhoff polytope, but among suitable subsets of the vertices of this canonical face and the corresponding representations of A.

In particular, the full vertex set of the minimal face need not satisfy the linear independence hypothesis of Theorem 5.2. By Carathéodory's theorem, A admits a representation as a convex combination of at most $(n-1)^2+1$ vertices, which may be chosen affinely independent. Such a representation is therefore a natural choice when one wishes to control the number m, although it need not minimize the largest invariant factor of the associated Gram matrix.

- *Linear versus affine independence.* Theorem 5.2 is stated in terms of linear independence because this is precisely the condition needed in its proof to ensure that the Gram matrix

$$M = (\langle P_i, P_j\rangle_{\mathrm{F}})_{i,j=1}^m$$

is positive definite and hence invertible. Carathéodory's theorem, on the other hand, naturally provides an affinely independent set of vertices. For families of permutation matrices, affine independence and linear independence are in fact equivalent.

Indeed, suppose that

$$\sum_{i=1}^{m} c_i P_i = 0.$$

Since every permutation matrix has every row sum equal to 1, summing the entries in any fixed row of this relation gives

$$\sum_{i=1}^{m} c_i = 0.$$

Thus every linear dependence relation among permutation matrices automatically satisfies the additional coefficient condition required for an affine dependence. Consequently, if $P_1, \ldots, P_m$ are affinely independent, then no nontrivial linear dependence among them can exist, and hence they are linearly independent. The converse holds for every family of vectors, since linear independence always implies affine independence. Therefore, for permutation matrices, the two notions coincide.
Accordingly, the linear independence assumption in Theorem 5.2 is the appropriate algebraic hypothesis for the Gram-matrix argument, while the affinely independent Birkhoff decomposition supplied by Carathéodory's theorem automatically satisfies this hypothesis.

Theorem 5.2 shows that the arithmetic complexity of an Erdős matrix is constrained by the invariant factors of the Gram matrix arising from a Birkhoff decomposition. In contrast with the graph-theoretic approach of Section 3, which relates denominators to spanning trees of support graphs, the present approach links denominator growth to the representation theory of the symmetric group and to the Smith normal form of an associated integer matrix. These two perspectives appear to capture different aspects of the arithmetic structure of Erdős matrices.

Theorem 5.2 provides an arithmetic bound on the denominator of an Erdős matrix in terms of the invariant factors of the associated Gram matrix. Combined with standard estimates from convex geometry and linear algebra, it yields a general upper bound for the arithmetic complexity function.

Although the resulting estimate is weaker than the graph-theoretic bound of Theorem 3.1, it arises from a fundamentally different mechanism involving Smith normal forms and the representation theory of the symmetric group.

Corollary 5.4. *There exists a constant $C > 0$ such that*

$$D_n \;\leq\; \exp\left(Cn^2 \log n\right)$$

for all $n \geq 1$.

Proof. Let A be an $n \times n$ Erdős matrix. By Carathéodory's theorem, A admits a Birkhoff decomposition

$$A = \sum_{i=1}^{m} x_i P_i,$$

with

$$m \leq (n-1)^2 + 1,$$

where the permutation matrices $P_1, \dots, P_m$ may be chosen affinely independent. By Remark 5.3, affine independence implies linear independence for permutation matrices. Hence the hypotheses of Theorem 5.2 are satisfied. Let

$$M_{ij} \;=\; \langle P_i, P_j \rangle_{\mathrm{F}}$$

be the associated Gram matrix, and let

$$d_1 \mid d_2 \mid \cdots \mid d_m$$

be its invariant factors.

By Theorem 5.2,

$$\operatorname{lcd}(A) \mid d_m \operatorname{lcd}\bigl(\|A\|_{\mathrm{F}}^2\bigr).$$

Since

$$d_1 \cdots d_m \;=\; \det(M),$$

we have

$$d_m \;\leq\; \det(M).$$

Each entry of M is bounded in absolute value by n, so every row has Euclidean norm at most $n\sqrt{m}$. Hadamard's inequality therefore gives

$$\det(M) \;\leq\; (n\sqrt{m})^m.$$

Let

$$\lambda = \|A\|_{\mathrm{F}}^2.$$

As shown in the proof of Theorem 5.2,

$$M\mathbf{x} \;=\; \lambda \mathbf{1}.$$

Since M is invertible, $\mathbf{x} = \lambda M^{-1}\mathbf{1}$, and summing the coordinates gives

$$1 \;=\; \mathbf{1}^{\mathsf{T}}\mathbf{x} \;=\; \lambda\, \mathbf{1}^{\mathsf{T}} M^{-1} \mathbf{1} \;=\; \lambda\, \frac{\mathbf{1}^{\mathsf{T}} \operatorname{adj}(M) \mathbf{1}}{\det(M)},$$

where $\mathbf{1}^\intercal M^{-1}\mathbf{1} > 0$ because M is positive definite. Hence

$$\lambda \;=\; \frac{\det(M)}{\mathbf{1}^\intercal \operatorname{adj}(M)\mathbf{1}},$$

and the denominator of λ divides

$$N \;=\; \mathbf{1}^\intercal \operatorname{adj}(M)\mathbf{1}.$$

Each entry of $\operatorname{adj}(M)$ is an $(m-1)\times(m-1)$ minor of M, and another application of Hadamard's inequality yields

$$|N| \;\le\; m^2(n\sqrt{m})^{m-1}.$$

Consequently,

$$\operatorname{lcd}(A) \;\le\; m^2(n\sqrt{m})^{2m-1}.$$

Since

$$m \;\le\; (n-1)^2+1 = O(n^2),$$

it follows that

$$\log \operatorname{lcd}(A) \;=\; O(n^2\log n).$$

Therefore

$$\operatorname{lcd}(A) \;\le\; \exp(Cn^2\log n)$$

for some absolute constant $C>0$ and all $n\ge 1$.

Taking the maximum over all Erdős matrices completes the proof. $\square$

6. Open questions

The results of this paper establish the bounds

$$\exp\bigl(c\sqrt{n\log n}\bigr) \;\le\; D_n \;\le\; \exp(2n\log n),$$

and show that

$$\mathcal{D}(N) \;\ge\; c\sqrt{\frac{N}{\log N}}.$$

The results obtained here provide the first general information on the arithmetic complexity of Erdős matrices. They establish nontrivial lower and upper bounds for the denominator sequence, yield a lower bound for the number of distinct denominators, and reveal connections with graph Laplacians and Smith normal forms. At the same time, they raise a number of natural questions, some of which we record below.

1. **Asymptotic growth of the largest denominator.**
   Determine the true asymptotic growth of D_n. In particular, what is the correct order of magnitude of $\log D_n$? The current lower and upper bounds remain far apart, and it is unclear whether either of them is close to optimal.
2. **Growth of the number of distinct denominators.**
   Does the estimate

$$\mathcal{D}(N) \;\geq\; c\sqrt{\frac{N}{\log N}}$$

   capture the correct order of growth of $\mathcal{D}(N)$? Equivalently, how rapidly can the set of denominators arising from Erdős matrices proliferate as the dimension increases?
3. **Arithmetic complexity of indecomposable Erdős matrices.**
   The lower-bound construction of this paper relies on block-diagonal sums and produces large denominators through least common multiples of pairwise coprime factors. This raises the question of whether comparable growth can occur in the indecomposable case.
   If

$$D_n^{\mathrm{ind}} := \max\Big\{\, \mathrm{lcd}(E) : E \text{ is an indecomposable } n \times n \text{ Erdős matrix} \Big\},$$

   what is the asymptotic behavior of D_n^{ind}?

Declaration of competing interest

The authors declare that they have no conflicts of interest.

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Data availability

No datasets were generated or analyzed during this work.

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