Sharp Spectral Inequalities for Graphs via Majorization and Schur-Convexity

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- We let $\|G\|_* := |\lambda_1| + |\lambda_2| + \cdots + |\lambda_n|$ be the *energy* of the graph G.

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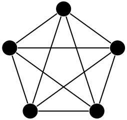

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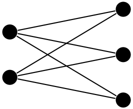

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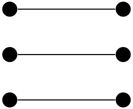

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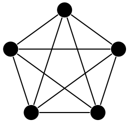

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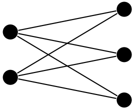

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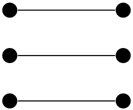

perfect matching graph M_n.

- Their spectrum are respectively

$$(n-1,-1,\dots,-1); \quad (\sqrt{rs},0,\dots,0,-\sqrt{rs}); \quad (1,\dots,1,-1,\dots,-1).$$

How *balanced* are these spectrum?

Question

Can we measure how *balanced* these spectrums are, and what does it reveal about the graph?

Majorization

Definition

Let $x_1^{\downarrow} \geq \cdots \geq x_n^{\downarrow}$ be the decreasing rearrangement of the entries of $\boldsymbol{x} = (x_1, \ldots, x_n)$, and let $\boldsymbol{x}^{\downarrow} = (x_1^{\downarrow}, \ldots, x_n^{\downarrow})$. Then $\boldsymbol{y}$ is *majorized* by $\boldsymbol{x}$, and we write $\boldsymbol{y} \prec \boldsymbol{x}$, if

$$\sum_{j=1}^{k} y_j^{\downarrow} \leq \sum_{j=1}^{k} x_j^{\downarrow}, \qquad k = 1, 2, \ldots, n,$$

with equality when $k = n$.

Example

$$\left(\tfrac{1}{n}, \ldots, \tfrac{1}{n}\right) \prec (a_1, \ldots, a_n) \prec (1, 0, \ldots, 0)$$

for all $a_j \geq 0$ $(1 \leq j \leq n)$ such that $\sum_{j=1}^{n} a_j = 1$.

Majorization order on the extremal graphs

Lemma (B.–Chávez–Sheng; 2026+)

Let G be a simple graph of order n with eigenvalues $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$. Let

- $\boldsymbol{\mu} = (n-1, -1, \ldots, -1)$;
- $\boldsymbol{\nu} = (\sqrt{rs}, 0, \ldots, 0, -\sqrt{rs})$ *for real numbers $r, s > 0$;*
- $\boldsymbol{w} = (1, \ldots, 1, -1, \ldots, -1)$ *if n is even.*

Then,

$$\frac{\lambda_1}{n-1}\boldsymbol{\mu} \prec \boldsymbol{\lambda} \prec |\lambda_n|\boldsymbol{\mu} \qquad \text{and} \qquad \frac{|\lambda_n|}{\sqrt{rs}}\boldsymbol{\nu} \prec \boldsymbol{\lambda} \prec \frac{\|G\|_*}{2\sqrt{rs}}\boldsymbol{\nu}.$$

Moreover, if G is bipartite and n is even, then

$$\frac{\|G\|_*}{n}\boldsymbol{w} \prec \boldsymbol{\lambda} \prec \lambda_1 \boldsymbol{w}.$$

Schur-convex Functions

Definitions

A function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is *Schur-convex* if $\varphi(\boldsymbol{x}) \leq \varphi(\boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ such that $\boldsymbol{x} \prec \boldsymbol{y}$.

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Properties

- Every function that is convex and symmetric is also Schur-convex.
- Every Schur-convex function is symmetric, but not necessarily convex.
- If f is symmetric and all first partial derivatives exist, then f is Schur-convex if and only if

$$(x_i - x_j)\left(\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_j}\right) \geq 0 \qquad \text{for all} \quad \boldsymbol{x} \in \mathbb{R}^n$$

  and all $1 \leq i, j \leq n$.

A family of inequalities

Theorem (B.–Chávez–Sheng; 2026+)

Let G be a simple graph of order n with eigenvalues $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$, and let φ be a positive homogeneous function on graphs which is Schur-convex relative to its eigenvalues. Then, for all integers $r, s > 0$ such that $r + s = n$,

$$\frac{\lambda_1}{n-1} \leq \frac{\varphi(G)}{\varphi(K_n)} \leq |\lambda_n| \leq \frac{\sqrt{rs}\varphi(G)}{\varphi(K_{r,s})} \leq \frac{\|G\|_*}{2}.$$

Moreover, if G is bipartite and n is even, then we further have

$$\frac{\|G\|_*}{n} \leq \frac{\varphi(G)}{\varphi(M_n)} \leq \lambda_1 \leq \frac{\sqrt{rs}\varphi(G)}{\varphi(K_{r,s})} \leq \frac{\|G\|_*}{2}.$$

Application to the Schatten p-norms

Corollary (B.–Chávez–Sheng; 2026+)

Let G be a simple graph of order n with eigenvalues $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$. Then, for all even $d \geq 2$,

$$\frac{\lambda_1^d}{(n-1)^d} \leq \frac{c_d(G)}{(n-1)^d + n - 1} \leq |\lambda_n|^d \leq \frac{c_d(G)}{2} \leq \frac{\|G\|_*^d}{2^d}.$$

Moreover, if G is bipartite and n is even, then we further have

$$\frac{\|G\|_*^d}{n^d} \leq \frac{c_d(G)}{n} \leq \lambda_1^d \leq \frac{c_d(G)}{2} \leq \frac{\|G\|_*^d}{2^d}.$$

The Random vector norms

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Then

$$\|A\|_{\boldsymbol{X},d} := \mathbb{E}\Big[|\lambda_1 X_1 + \lambda_2 X_2 + \cdots + \lambda_n X_n|^d\Big]^{\frac{1}{d}}$$

are matrix norms on the space of Hermitian matrices.

Properties of these norms

Theorem (Chávez–Garcia–Hurley; 2023)

Under the previous hypothesis,

1. $\|UZU^*\|_{\boldsymbol{X},d} = \|Z\|_{\boldsymbol{X},d}$ *for any unitary matrix* U*;*
2. $\|Z\|_{\boldsymbol{X},d}$ *is continuous relative to* d*;*
3. $\|Z\|_{\boldsymbol{X},d_1} \leq \|Z\|_{\boldsymbol{X},d_2}$ *if* $d_1 \leq d_2$*;*

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4. $\|Z_1\|_{\boldsymbol{X},d} \leq \|Z_2\|_{\boldsymbol{X},d}$ *if* $\boldsymbol{\lambda}(Z_1) \prec \boldsymbol{\lambda}(Z_2)$*;*

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3. $\|Z\|_{\boldsymbol{X},d_1} \leq \|Z\|_{\boldsymbol{X},d_2}$ *if* $d_1 \leq d_2$*;*
4. $\|Z_1\|_{\boldsymbol{X},d} \leq \|Z_2\|_{\boldsymbol{X},d}$ *if* $\boldsymbol{\lambda}(Z_1) \prec \boldsymbol{\lambda}(Z_2)$*;*
5. *If* $d \geq 2$ *is even,* X *has the cumulant sequence* $\kappa_1, \kappa_2, \ldots,$ *and* G *is a graph of order* n *with* c_k *closed walks of length* k*, then*

$$\|G\|_{X,d}^d = \sum_{\boldsymbol{\pi} \vdash d} \frac{\kappa_{\boldsymbol{\pi}}}{y_{\boldsymbol{\pi}}} c_{\boldsymbol{\pi}},$$

where $y_{\boldsymbol{\pi}} := \prod_{i \geq 1} (i!)^{m_i} m_i!$, $\kappa_{\boldsymbol{\pi}} := \kappa_{\pi_1} \cdots \kappa_{\pi_\ell}$ *and* $c_{\boldsymbol{\pi}} := c_{\pi_1} \cdots c_{\pi_\ell}$.

Optimal bounds when $d \leq 4$

Theorem (B.–Chávez–Sheng; 2026+)

Let G be a simple graph of order n with eigenvalues $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$. Then

1. $\lambda_1^4 \leq \frac{c_4}{1+(n-1)^{-3}}$,
2. $8c_4 + 6(c_2^2 - 2c_4)^+ \leq \|G\|_*^4$,
3. $\frac{3c_2^2-2c_4}{n(n-1)(n^2+3n-6)} \leq \lambda_n^4 \leq \frac{4c_4-3(2c_4-c_2^2)^+}{8}$,

Moreover, if G is bipartite and n is even, then we further have

1. $\frac{c_4}{2\lfloor n/2 \rfloor} \leq \lambda_1^4 \leq \frac{4c_4-3(2c_4-c_2^2)^+}{8}$,
2. $8c_4 + 6(c_2^2 - 2c_4)^+ \leq \|G\|_*^4 \leq \frac{n^3(3c_2^2-2c_4)}{3n-2}$.

The inequalities are best possible among the random vector norms with $d \in \{2, 4\}$.

Some applications

Corollary (B.–Chávez–Sheng; 2026+)

Let G be a connected graph of order n with m edges and q squares. Then the extremal eigenvalues of G satisfy the following sharp bounds:

$$\frac{4\big(m(3m+1)-n(2m-n+1)-4q\big)}{n(n-1)(n^2+3n-6)} \le \lambda_n^4 \le \lambda_1^4 \le \frac{8q-2m+2n(2m-n+1)}{1+(n-1)^{-3}}.$$

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Corollary (B.–Chávez–Sheng; 2026+)

Let B be a square-free bipartite graph of order n with m edges. The spectral radius of B satisfies the following bound, which is attained by S_n:

$$\lambda_1 \leq \sqrt[4]{n(n-1)-m}.$$

Standard exponential distribution

Theorem (B.–Chávez–Sheng; 2026+)

Suppose $d \geq 2$ is even and B is a bipartite graph of order n with m edges and spectral radius λ_1. Then we have the bound

$$c_d \geq \left(\frac{2m}{\lambda_1^2} + d - \frac{d}{(\frac{d}{2})!} \left(\frac{m}{\lambda_1^2} \right)^{(\frac{d}{2})} \right) \lambda_1^d,$$

which is attained by any complete bipartite graph.

Uniform distribution on $[-1, 1]$

Theorem (B.–Chávez–Sheng; 2026+)

Let G be a simple graph of order n with m edges. If G has either (a) no triangle or (b) no square and $m \geq 9$, then

$$4c_6 + 34m^3 \geq 21mc_4.$$

Moreover, equality is realized by any complete bipartite graph.

Open Questions

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(iv) Find whether or not this technique can be used by beginning with spectrum other than K_n, $K_{r,s}$ and M_n.

(v) Find ways, if possible, to use this technique on connected graphs to get sharper results.

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