The norm of infinite doubly stochastic matrices

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Doubly stochastic matrices

Definition

A square matrix is *doubly stochastic* if:

- nonnegative coefficients;
- row sums $= 1$;
- column sums $= 1$.

The set of $n \times n$ doubly stochastic matrices is denoted by Ω_n.

Example

$$D = \begin{bmatrix} 0.1 & 0.3 & 0.6 \\ 0.4 & 0.2 & 0.4 \\ 0.5 & 0.5 & 0 \end{bmatrix}.$$

Birkhoff's theorem

Theorem (Birkhoff; 1946)

The set Ω_n of $n \times n$ doubly stochastic matrices is the convex hull of the $n \times n$ permutation matrices. Furthermore, the permutation matrices are precisely the extreme points of Ω_n.

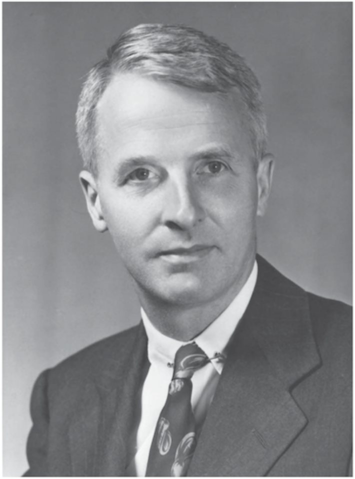

Figure: Garrett Birkhoff

Majorization

Definition

Let $x_1^{\downarrow} \geq \cdots \geq x_n^{\downarrow}$ be the decreasing rearrangement of the entries of $x = (x_1, \ldots, x_n)$. Then y is *majorized* by x, and we write $y \prec x$, if

$$\sum_{j=1}^{k} y_j^{\downarrow} \leq \sum_{j=1}^{k} x_j^{\downarrow}, \qquad k = 1, 2, \ldots, n,$$

with equality when $k = n$.

Example

$$\left(\frac{1}{n}, \ldots, \frac{1}{n}\right) \prec (a_1, \ldots, a_n) \prec (1, 0, \ldots, 0)$$

for all $a_j \geq 0$ $(1 \leq j \leq n)$ such that $\sum_{j=1}^{n} a_j = 1$.

The Hardy, Littlewood, Pólya, Schur, and Horn theorem

Theorem (Schur, 1923; HLP, 1929; Horn, 1954)

Let $x, y \in \mathbb{R}^n$. The following statements are equivalent:

1. $y = Dx$ *for some doubly stochastic matrix* D;
2. $y \prec x$;
3. $\sum_{i=1}^{n} \phi(y_i) \leq \sum_{i=1}^{n} \phi(x_i)$ *for every convex function* ϕ;
4. *There exists a Hermitian matrix whose main diagonal entries are given by* y *and whose eigenvalues are given by* x.

The Motivation

Question

Can we generalize *majorization* to infinite dimension?

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(Partial) answer

Yes, if D is well-defined as an operator from ℓ^p to ℓ^p.

The ℓ^p operator norm in finite dimension

Theorem (B.; 2021)

For all $1 \leq p \leq \infty$ and all finite doubly stochastic matrix D, we have

$$\|D\|_{\ell^p \to \ell^p} = 1.$$

Reminder

By definition,

$$\|D\|_{\ell^p \to \ell^p} := \sup_{\boldsymbol{x} \neq \boldsymbol{0}} \frac{\|D\boldsymbol{x}\|_p}{\|\boldsymbol{x}\|_p}.$$

Short proof

Upper bound: By Birkhoff, there exists some $\alpha_i \geq 0$ and some permutation matrices such that $\sum_i \alpha_i = 1$ and $\sum_i \alpha_i P_i = D$.

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$$\implies \quad \|D\|_{\ell^p \to \ell^p} = \left\| \sum_i \alpha_i P_i \right\|_{\ell^p \to \ell^p} \leq \sum_i \alpha_i \|P_i\|_{\ell^p \to \ell^p} = \sum_i \alpha_i = 1.$$

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Lower bound: Every doubly stochastic matrix satisfy $D\boldsymbol{e}_n = \boldsymbol{e}_n$, where $\boldsymbol{e}_n$ is the all-ones vector.

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Lower bound: Every doubly stochastic matrix satisfy $D\boldsymbol{e}_n = \boldsymbol{e}_n$, where $\boldsymbol{e}_n$ is the all-ones vector.

$$\implies \qquad \|D\|_{\ell^p \to \ell^p} = \sup_{\boldsymbol{x} \neq \boldsymbol{0}} \frac{\|D\boldsymbol{x}\|_p}{\|\boldsymbol{x}\|_p} \geq \frac{\|D\boldsymbol{e}_n\|_p}{\|\boldsymbol{e}_n\|_p} = \frac{\|\boldsymbol{e}_n\|_p}{\|\boldsymbol{e}_n\|_p} = 1.$$

The ℓ^p operator norm in infinite dimension

Theorem (Eshkaftaki; 2021)

For all $1 \leq p \leq \infty$ and all infinite doubly stochastic matrix D, we have

$$\|D\|_{\ell^p \to \ell^p} \leq 1.$$

Moreover, we have

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Question

If $1 < p < \infty$, are there infinite doubly stochastic matrices such that

$$\|D\|_{\ell^p \to \ell^p} < 1\,?$$

An enlightening example

Theorem (Eshkaftaki; 2021)

Let $1 < p < \infty$ and let $D_k = S_k + S_k^*$, where

$$S_k := \begin{bmatrix} \underbrace{\frac{1}{k+1} \quad \cdots \quad \frac{1}{k+1}}_{k \text{ times}} & 0 \quad \cdots \quad 0 & \cdots \\ 0 \quad \cdots \quad 0 & \underbrace{\frac{1}{k+1} \quad \cdots \quad \frac{1}{k+1}}_{k \text{ times}} & \cdots \\ 0 \quad \cdots \quad 0 & 0 \quad \cdots \quad 0 & \cdots \\ \vdots \qquad\quad \vdots & \vdots \qquad\quad \vdots & \ddots \end{bmatrix}.$$

Then D_k is doubly stochastic and

$$\|D_k\|_{\ell^p \to \ell^p} \leq \|S_k\|_{\ell^p \to \ell^p} + \|S_k^*\|_{\ell^p \to \ell^p} \leq \frac{k^{1/p} + k^{1-1/p}}{k+1} \longrightarrow 0.$$

Main Question

Can we characterize the infinite doubly stochastic matrices D for which $\|D\|_{\ell^p \to \ell^p} = 1$?

An important observation

Lemma (B., Mashreghi, Vo; 2026+)

Let D be an infinite doubly stochastic matrix and let $1 < p < \infty$. Then

$$\|D\|_{\ell^p \to \ell^p} = 1 \quad \Longleftrightarrow \quad \|D\|_{\ell^2 \to \ell^2} = 1.$$

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Proof.

Apply Riesz–Thorin interpolation theorem, which states that if $1 \leq p_0 < p_1 \leq \infty$ and p_θ satisfies $\frac{1}{p_\theta} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1}$ for some $0 < \theta < 1$, then

$$\|D\|_{\ell^{p_\theta}(I) \to \ell^{p_\theta}(I)} \leq \|D\|^{\theta}_{\ell^{p_0}(I) \to \ell^{p_0}(I)} \|D\|^{1-\theta}_{\ell^{p_1}(I) \to \ell^{p_1}(I)}.$$

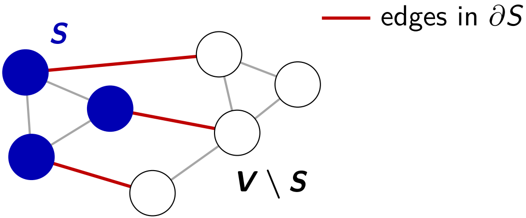
$\square$

The Edge Boundary

Definition

Let $G = (V, E)$ be a graph with vertex set V and edge set E and let $S \subseteq V$ be a subset of vertices. The *edge boundary* of S is the set

$$\partial S := \{\{x, y\} \in E : x \in S,\ y \in V \setminus S\}.$$



The Cheeger Constant

Definition

The *Cheeger constant* of a graph G is defined as

$$\Phi(G) := \inf\left\{\frac{|\partial S|}{|S|} \;:\; S \subseteq V,\, 0 < |S| \leq \tfrac{1}{2}|V|\right\}$$

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If A is the adjacency matrix of G, then

$$\Phi(A) := \Phi(G) = \inf_{S \subseteq V} \frac{1}{|S|} \sum_{i \in S} \sum_{j \notin S} a_{ij}.$$

An application to infinite doubly stochastic matrices

Lemma (B., Mashreghi, Vo; 2026+)

If D is an infinite doubly stochastic matrix, then

$$\frac{1}{2}\left(1 - \|D\|^2_{\ell^2\to\ell^2}\right)^2 \leq \Phi(D^*D) \leq \sqrt{1 - \|D\|^4_{\ell^2\to\ell^2}}$$

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Corollary (B., Mashreghi, Vo; 2026+)

If D is an infinite doubly stochastic matrix, then

$$\|D\|_{\ell^2\to\ell^2} = 1 \quad \Longleftrightarrow \quad \Phi(D^*D) = 0.$$

An Elegant Interpretation

Remark

For any (infinite) doubly stochastic matrix D, we have

$$\Phi(D) = 1 - \Theta(D),$$

where

$$\Theta(D) := \sup_{S \subseteq V} \frac{1}{|S|} \sum_{i,j \in S} d_{ij}$$

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Theorem (B., Mashreghi, Vo; 2026+)

If D is an infinite doubly stochastic matrix, then

$$\|D\|_{\ell^2 \to \ell^2} = 1 \quad \Longleftrightarrow \quad \Theta(D^* D) = 1.$$

A Formula for the Norm

Theorem (B., Mashreghi, Vo; 2026+)

If D is an infinite doubly stochastic matrix, then

$$\|D\|_{\ell^2 \to \ell^2} = \sup_{n \geq 1} \Theta\big((D^* D)^n\big)^{1/n}.$$

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$$\|D\|_{\ell^2 \to \ell^2} = \sup_{n \geq 1} \Theta\big((D^* D)^n\big)^{1/n}.$$

Remark

One may hope to replace $\Theta(D^* D)$ in the above results by $\Theta(D)$.

However, there exists an example of a doubly stochastic matrix such that

$$\Theta(D) \leq \frac{1}{2} \qquad \& \qquad \Theta(D^* D) = 1.$$

References

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An Explicit Example

Let $T = (I, E)$ be an infinite 4-regular tree. Choose an orientation of the edges of T such that every vertex has exactly two outgoing edges and two incoming edges.
Then the matrix D defined by

$$d_{ij} = \begin{cases} \frac{1}{2}, & \text{if } i \to j \text{ is an oriented edge of } T, \\ 0, & \text{otherwise} \end{cases}$$

has the desired properties.