New Results on the Doubly Stochastic Inverse Eigenvalue Problem

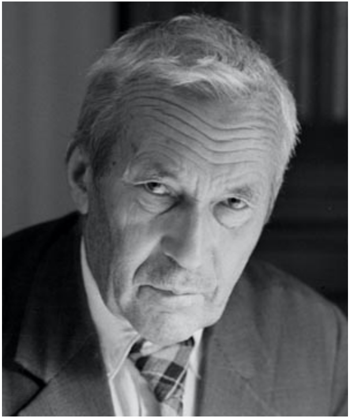
Ludovick Bouthat

Université Laval

May 21, 2026

A Spectral Problem

• **1938 :** Kolmogorov propose the problem of characterizing the set Ω_n of all the possible eigenvalues for $n \times n$ stochastic matrices.

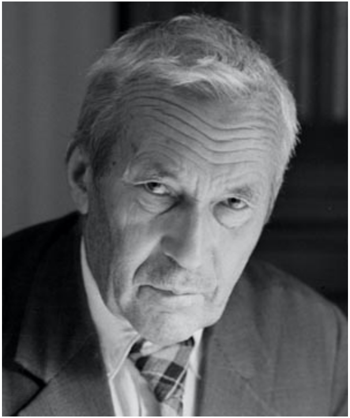


Andreï Kolmogorov

A Spectral Problem

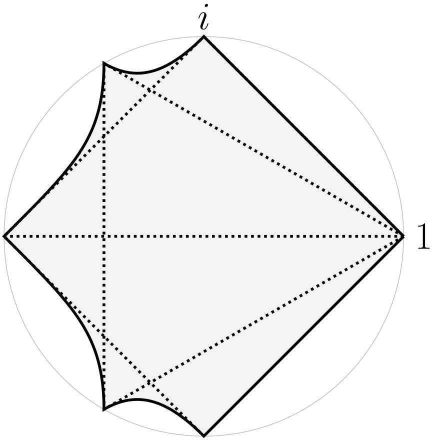
- **1938 :** Kolmogorov propose the problem of characterizing the set Ω_n of all the possible eigenvalues for $n \times n$ stochastic matrices.

- **1951 :** Karpelevič obtained a complete description of Ω_n for all $n \geq 1$.

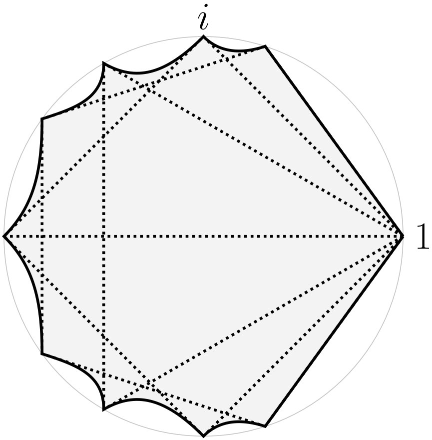


Andreï Kolmogorov

The Karpelevič Regions



The region Ω_4

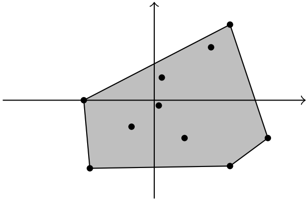


The region Ω_5

Proof Idea

Theorem (Dmitriev, Dynkin; 1946)

Let $\vec{z} \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$. Define $K_1 := \mathrm{Conv}\{z_1, z_2, \ldots, z_n\}$. Then there exists an $n \times n$ stochastic matrix S such that $S\vec{z} = \lambda\vec{z}$ if and only if $\lambda K_1 \subseteq K_1$.



$K_1 = \mathrm{Conv}\{z_1, z_2, \ldots, z_n\}$

Main Problem

Question (Mirsky; 1963)

For which numbers $\lambda \in \mathbb{C}$ is there an $n \times n$ *doubly stochastic* matrix D such that λ is an eigenvalue of D? That is, characterize the set

$$\omega_n := \left\{ \lambda \in \mathbb{C} : \exists D \in \mathbb{D}_n \text{ such that } \lambda \in \sigma(D) \right\} = \bigcup_{D \in \mathbb{D}_n} \sigma(D).$$

Reminder

- D is doubly stochastic if both D and D^t are stochastic.
- $\mathbb{D}_n :=$ The set of $n \times n$ doubly stochastic matrices.

Properties of ω_n

Theorem

We have the set inclusion

$$\omega_n \subseteq \Omega_n \subseteq \overline{\mathbb{D}}.$$

Moreover, ω_n is star-shaped from 0.

Properties of ω_n

Theorem

We have the set inclusion

$$\omega_n \subseteq \Omega_n \subseteq \overline{\mathbb{D}}.$$

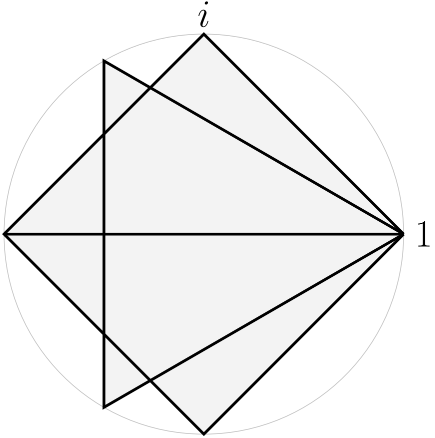
Moreover, ω_n is star-shaped from 0.

Theorem (Perfect, Mirsky; 1965)

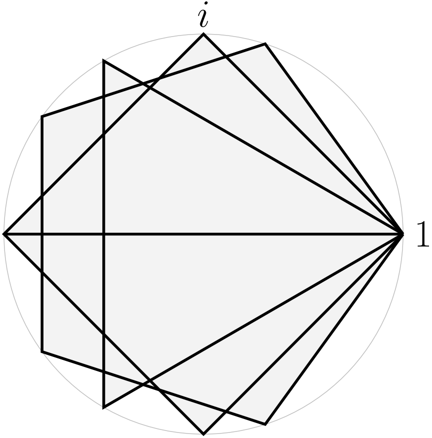
Let $\zeta_k := e^{2\pi i/k}$ and $\Pi_k := \mathrm{Conv}\{1, \zeta_k, \zeta_k^2, \ldots, \zeta_k^{k-1}\}$. Then, for $n \geq 1$,

$$\Pi_1 \cup \Pi_2 \cup \cdots \cup \Pi_n \subseteq \omega_n.$$

Some Examples



The region $\cup_{k=1}^{4} \Pi_k$



The region $\cup_{k=1}^{5} \Pi_k$

The Perfect–Mirsky Conjecture

Conjecture (Perfect, Mirsky; 1965)

$$\omega_n = \bigcup_{k=1}^{n} \Pi_k =: PM_n.$$

The Perfect–Mirsky Conjecture

Conjecture (Perfect, Mirsky; 1965)

$$\omega_n = \bigcup_{k=1}^{n} \Pi_k =: PM_n.$$

- $n = 1$: True
- $n = 2$: True

The Perfect–Mirsky Conjecture

Conjecture (Perfect, Mirsky; 1965)

$$\omega_n = \bigcup_{k=1}^{n} \Pi_k =: PM_n.$$

- $n = 1$: True
- $n = 2$: True
- $n = 3$: True (Perfect, Mirsky; 1965).

The Perfect–Mirsky Conjecture

Conjecture (Perfect, Mirsky; 1965)

$$\omega_n = \bigcup_{k=1}^{n} \Pi_k =: PM_n.$$

- $n = 1$: True
- $n = 2$: True
- $n = 3$: True (Perfect, Mirsky; 1965).
- $n = 4$: True (Levick, Pereira, Kribs; 2015).

The Perfect–Mirsky Conjecture

Conjecture (Perfect, Mirsky; 1965)

$$\omega_n = \bigcup_{k=1}^{n} \Pi_k =: PM_n.$$

- $n = 1$: True
- $n = 2$: True
- $n = 3$: True (Perfect, Mirsky; 1965).
- $n = 4$: True (Levick, Pereira, Kribs; 2015).
- $n = 5$: *False* (Mashreghi, Rivard; 2007).

The Perfect–Mirsky Conjecture

Conjecture (Perfect, Mirsky; 1965)

$$\omega_n = \bigcup_{k=1}^{n} \Pi_k =: PM_n.$$

- $n = 1$: True
- $n = 2$: True
- $n = 3$: True (Perfect, Mirsky; 1965).
- $n = 4$: True (Levick, Pereira, Kribs; 2015).
- $n = 5$: *False* (Mashreghi, Rivard; 2007).
- $n \geq 6$: No known results.

An Important Particular Case

Definition

A vector $\vec{z} \in \mathbb{C}^n$ is *convexly independent* if

$$K_1 = \mathrm{Conv}\{z_1, z_2, \ldots, z_n\}$$

has exactly n extreme points.

An Important Particular Case

Definition

A vector $\vec{z} \in \mathbb{C}^n$ is *convexly independent* if

$$K_1 = \mathrm{Conv}\{z_1, z_2, \ldots, z_n\}$$

has exactly n extreme points.

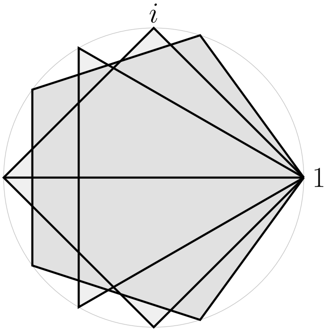
Conjecture (Levick, Pereira, Kribs; 2015)

If $\lambda \in \mathbb{C}$ is an eigenvalue of an $n \times n$ doubly stochastic matrix associated to a convexly independent eigenvector, then $\lambda \in PM_n$.

A Positive Result

Theorem (B., Mashreghi, Morneau-Guérin; 2026+)

If $\lambda \in \mathbb{C}$ is an eigenvalue of an $n \times n$ doubly stochastic matrix associated to a convexly independent eigenvector, then $\lambda \in \Pi_n \subsetneq PM_n$.



An Essential Implication

Definition

Let $\vec{z} \in \mathbb{C}^n$. We define the polygons

$$K_1 := \mathrm{Conv}\{z_1, z_2, \ldots, z_n\},$$

$$K_2 := \mathrm{Conv}\left\{\frac{z_i + z_j}{2} : i \neq j\right\},$$

$$\vdots$$

$$K_m := \mathrm{Conv}\left\{\frac{z_{i_1} + z_{i_2} + \cdots + z_{i_m}}{m} : i_j \neq i_k \text{ for all } j \neq k\right\},$$

$$\vdots$$

$$K_n := \mathrm{Conv}\left\{\frac{z_1 + z_2 + \cdots + z_n}{n}\right\}.$$

An Essential Implication

Theorem (B., Mashreghi, Morneau-Guérin; 2026+)

If $\lambda \in \omega_n$, then there exists $\vec{z} \in \mathbb{C}^n$ such that

$$\lambda K_m \subseteq K_m \quad \text{for all} \quad m = 1, 2, \ldots, n.$$

A Difficult Problem

Theorem (Dmitriev, Dynkin; 1946)

$\lambda \in \Omega_n$ if and only if there exists $\vec{z} \in \mathbb{C}^n$ such that

$$\lambda K_1 \subseteq K_1.$$

Theorem (B., Mashreghi, Morneau-Guérin; 2026+)

$\lambda \in \omega_n$ only if there exists $\vec{z} \in \mathbb{C}^n$ such that

$$\lambda K_m \subseteq K_m \quad \text{for all } m = 1, 2, \ldots, n.$$

Reformulation with Circulant Matrices

Observation : If $\vec{z}$ is convexly independent, then $K_2, K_3, \ldots,$ have a simple form.

Reformulation with Circulant Matrices

Observation : If $\vec{z}$ is convexly independent, then $K_2, K_3, \ldots,$ have a simple form.

Proposition (B., Mashreghi, Morneau-Guérin; 2026+)

If $\vec{z} \in \mathbb{C}^n$ is convexly independent with the z_j in clockwise order, then the extreme points of K_m are given in clockwise order by $T_m \vec{z}$, where T_m is the circulant matrix

$$T_m := \operatorname{circ}\big(\underbrace{1, \ldots, 1}_{m}, \underbrace{0, \ldots, 0}_{n-m}\big).$$

Maximal Eigenvectors

Definition

An eigenpair $(re^{i\theta}, \vec{z})$ of an $n \times n$ doubly stochastic matrix, with $\vec{z}$ convexly independent, is called *maximal* if $(r + \varepsilon)e^{i\theta}$ is not an eigenvalue of any $n \times n$ doubly stochastic matrix with a convexly independent eigenvector.

Maximal Eigenvectors

Definition

An eigenpair $(re^{i\theta}, \vec{z})$ of an $n \times n$ doubly stochastic matrix, with $\vec{z}$ convexly independent, is called *maximal* if $(r+\varepsilon)e^{i\theta}$ is not an eigenvalue of any $n \times n$ doubly stochastic matrix with a convexly independent eigenvector.

Conjecture (B., Mashreghi, Morneau-Guérin; 2026+)

If $(\lambda, \vec{z})$ *is a maximal eigenpair, then*

$$T_2\vec{z} = \left(\frac{z_1+z_2}{2}, \frac{z_2+z_3}{2}, \dots, \frac{z_{n-1}+z_n}{2}, \frac{z_n+z_1}{2}\right)$$

is also a maximal eigenvector (associated to λ*).*

A Dynamical Approach

Corollary (B., Mashreghi, Morneau-Guérin; 2026+)

If $(\lambda, \vec{z})$ is a maximal eigenpair, then $\rho^k T_2^k \vec{z}$ is also a maximal eigenpair (associated to λ) for all $\rho > 0$ and all $k \in \mathbb{N}$.

A Dynamical Approach

Corollary (B., Mashreghi, Morneau-Guérin; 2026+)

If $(\lambda, \vec{z})$ is a maximal eigenpair, then $\rho^k T_2^k \vec{z}$ is also a maximal eigenpair (associated to λ) for all $\rho > 0$ and all $k \in \mathbb{N}$.

Consequence : Letting $k \to \infty$ in the corollary and using a continuity argument, we may suppose without loss of generality that $\vec{z}$ satisfy

$$z_j = \alpha e^{\frac{2\pi ij}{n}} + \beta e^{\frac{-2\pi ij}{n}}, \qquad j = 1, 2, \ldots, n,$$

for some $\alpha, \beta \in \mathbb{C}$.

A Dynamical Approach

Corollary (B., Mashreghi, Morneau-Guérin; 2026+)

If $(\lambda, \vec{z})$ is a maximal eigenpair, then $\rho^k T_2^k \vec{z}$ is also a maximal eigenpair (associated to λ) for all $\rho > 0$ and all $k \in \mathbb{N}$.

Consequence : Letting $k \to \infty$ in the corollary and using a continuity argument, we may suppose without loss of generality that $\vec{z}$ satisfy

$$z_j = \alpha e^{\frac{2\pi ij}{n}} + \beta e^{\frac{-2\pi ij}{n}}, \qquad j = 1, 2, \ldots, n,$$

for some $\alpha, \beta \in \mathbb{C}$.

Maximizing $|\lambda|$ for the eigenvectors of this form yield the desired result. □

Heuristic for the cases $n \geq 6$

- The question can be reformulated as a constrained optimization problem.

Heuristic for the cases $n \geq 6$

- The question can be reformulated as a constrained optimization problem.

$$\begin{cases} \text{Number of variables} \approx 2n \\ \text{Number of constraints} \approx n\lfloor \frac{n}{2} \rfloor \end{cases}$$

Heuristic for the cases $n \geq 6$

- The question can be reformulated as a constrained optimization problem.

$$\begin{cases} \text{Number of variables} \approx 2n \\ \text{Number of constraints} \approx n\lfloor \frac{n}{2} \rfloor \end{cases} \implies \begin{cases} \text{variable} \geq \text{constraint} & \text{if } n \leq 5 \\ \text{variable} < \text{constraint} & \text{if } n > 5 \end{cases}$$

Consequence : Too many constraints force the solution into the local minimum that is the Perfect–Mirsky regions for $n > 5$.

Explanation for the cases $n \leq 4$

- If $n \leq 4 \implies K_1$ has $m \leq 3$ extreme points or $m = n = 4$ extreme points.

Explanation for the cases $n \leq 4$

- If $n \leq 4 \implies K_1$ has $m \leq 3$ extreme points or $m = n = 4$ extreme points.
- If $m \leq 3$, then

$$\lambda \in \omega_n \quad \implies \quad \lambda K_1 \subseteq K_1 \quad \implies \quad \lambda \in \Omega_m = \omega_m.$$

Explanation for the cases $n \leq 4$

- If $n \leq 4 \implies K_1$ has $m \leq 3$ extreme points or $m = n = 4$ extreme points.
- If $m \leq 3$, then

$$\lambda \in \omega_n \quad \implies \quad \lambda K_1 \subseteq K_1 \quad \implies \quad \lambda \in \Omega_m = \omega_m.$$

- If $m = n = 4$, then our result shows that $\lambda \in \Pi_4$.

Consequence for the Case $n = 5$

Theorem (B., Mashreghi, Morneau-Guérin; 2026+)

If $(\lambda, \vec{z})$ *is an eigenpair of a* 5×5 *doubly stochastic matrix which* <u>*is not*</u> *in* PM_5*, then* $K_1 = \mathrm{Conv}\{z_1, z_2, z_3, z_4, z_5\}$ *has exactly 4 extreme points.*

An Optimization Approach

Theorem (B., Mashreghi, Morneau-Guérin; 2026+)

If $\lambda \in \omega_n$, then there exists $\vec{z} \in \mathbb{C}^n$ such that

$$\lambda K_m \subseteq K_m \quad \text{for all} \quad m = 1, 2, \ldots, n.$$

Corollary (B., Mashreghi, Morneau-Guérin; 2026+)

Let $\vec{z} \in \mathbb{C}^n$. Then

$$r \leq \min_{\substack{1 \leq k < n \\ \phi \in [0,\pi)}} \frac{\sum_{j=1}^{k} \mathrm{Re}\left(e^{i\theta} z_j\right)^{\downarrow}}{\sum_{j=1}^{k} \mathrm{Re}\left(e^{i(\phi+\theta)} z_j\right)^{\downarrow}}$$

if and only if $re^{i\theta} K_m \subseteq K_m$ for all $m = 1, 2, \ldots, n$.

An Approach for $n \geq 6$

Corollary (B., Mashreghi, Morneau-Guérin; 2026+)

Let $R_n(\theta)$ be the polar equation of ∂PM_n. If

$$R_n(\theta) = \max_{\vec{z}\in\mathbb{C}^n} \min_{\substack{1\leq k<n \\ \phi\in[0,\pi)}} \frac{\sum_{j=1}^{k} \mathrm{Re}\left(e^{i\theta} z_j\right)^{\downarrow}}{\sum_{j=1}^{k} \mathrm{Re}\left(e^{i(\phi+\theta)} z_j\right)^{\downarrow}},$$

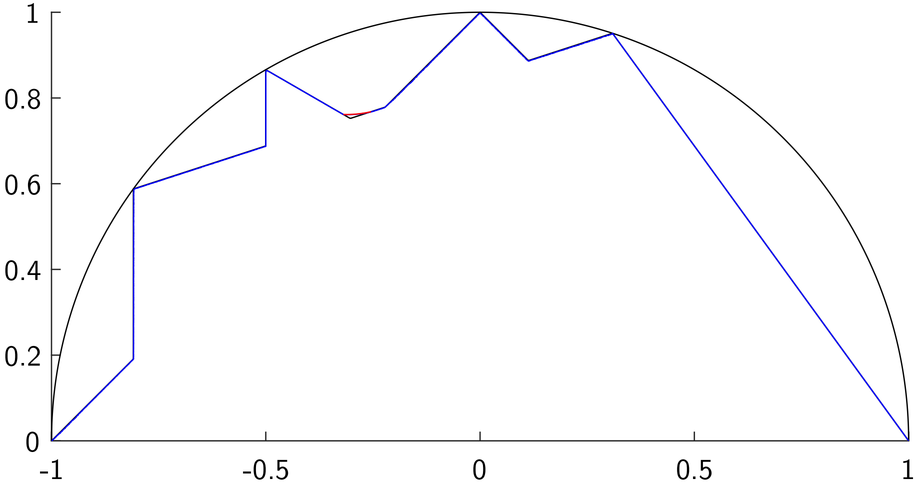
then $\omega_n = PM_n$.

Remark

The polar equation of ∂PM_n is given by

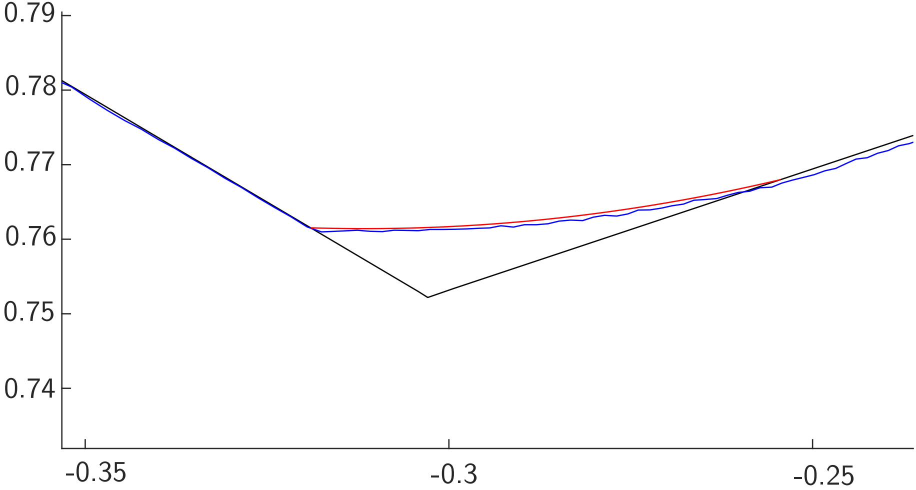
$$R_n(\theta) = \max_{k\in\{3,\ldots,n\}} \frac{\cos\left(\frac{\pi}{k}\right)}{\cos\left(\frac{\pi(2\lceil k\theta/2\pi\rceil - 1)}{k} - \theta\right)}, \qquad n \geq 4.$$

Numerical Computations



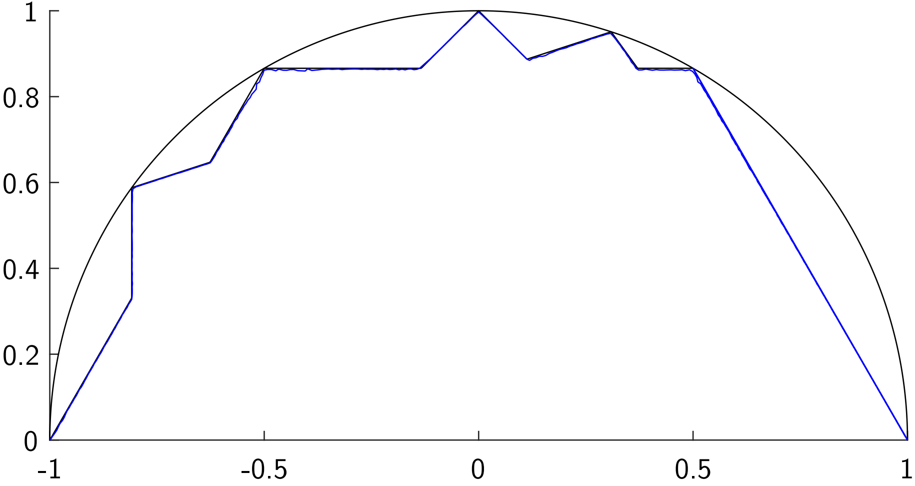
Approximation of the optimal radius for $n = 5$

Numerical Computations



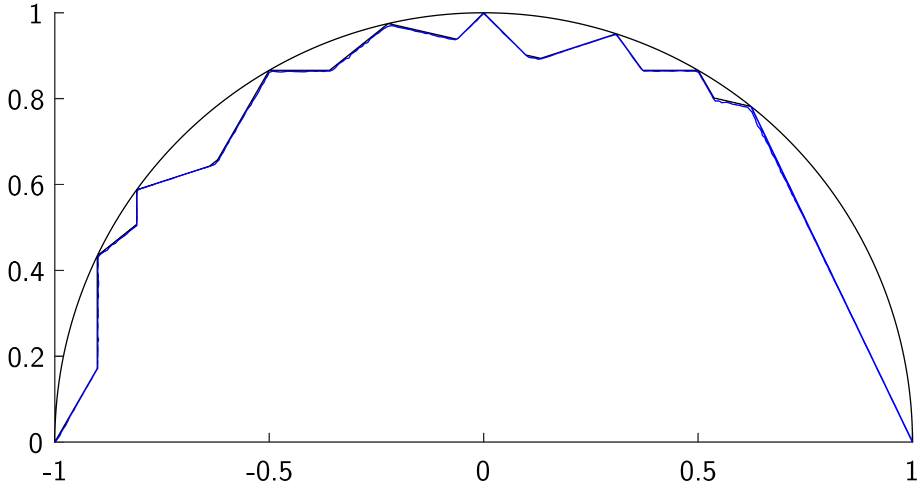
Zoom of the exceptional region for $n = 5$

Numerical Computations



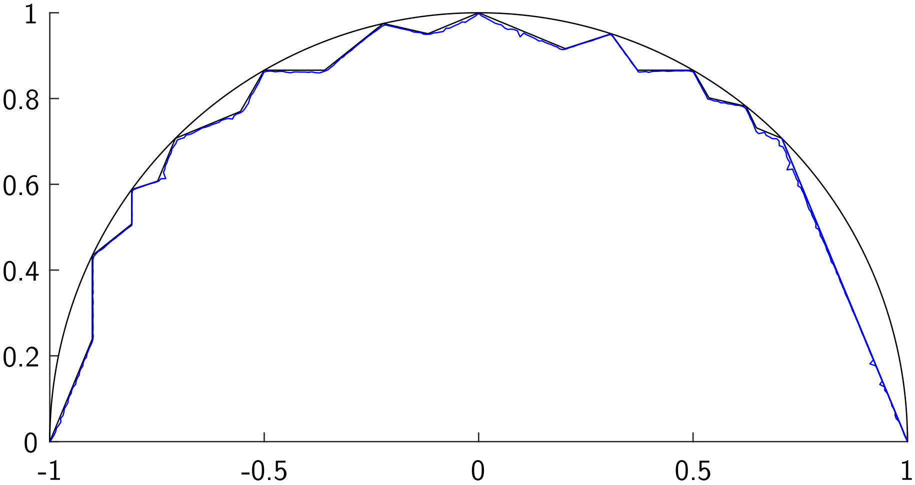
Approximation of the optimal radius for $n = 6$

Numerical Computations



Approximation of the optimal radius for $n = 7$

Numerical Computations



Approximation of the optimal radius for $n = 8$

The Exception when $n = 5$

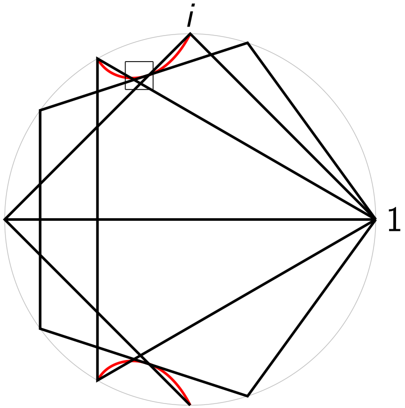
Theorem (Mashreghi, Rivard; 2007)

The 5×5 *doubly stochastic matrices*

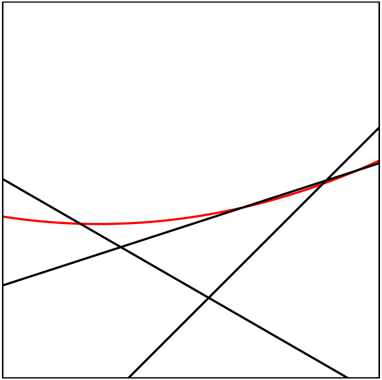
$$t \begin{pmatrix} 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} \\ \mathbf{1} & 0 & 0 & 0 & 0 \end{pmatrix} + (1-t) \begin{pmatrix} 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 & 0 \end{pmatrix}$$

have eigenvalues outside of the region $\bigcup_{k=1}^{5} \Pi_k$ *for* $t \in [0.47053, 0.54900]$.

The Exceptional Curve



The Perfect–Mirksy region $\bigcup_{k=1}^{5} \Pi_k$ and the Mashreghi–Rivard exceptional curve.



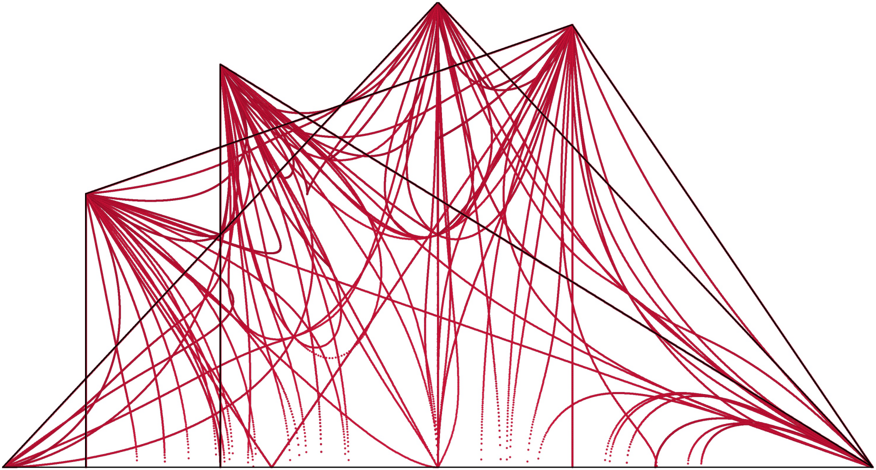
Zoom on the exceptional curve around $[-0.35, -0.2] \times [0.7, 0.85]$.

Conjectures of Harlev, Johnson and Lim

Conjecture (Boundary conjecture; 2022)

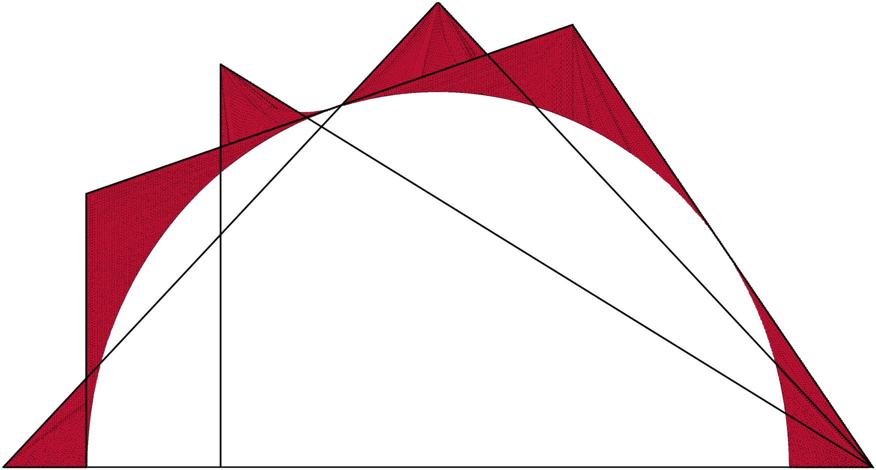
For $n \geq 1$, the pairs of permutation matrices characterize the boundary of ω_n. That is, each point of $\partial\omega_n$ is an eigenvalue of a convex combination of at most two permutation matrices.

The Numerical Computations of Harlev–Johnson–Lim



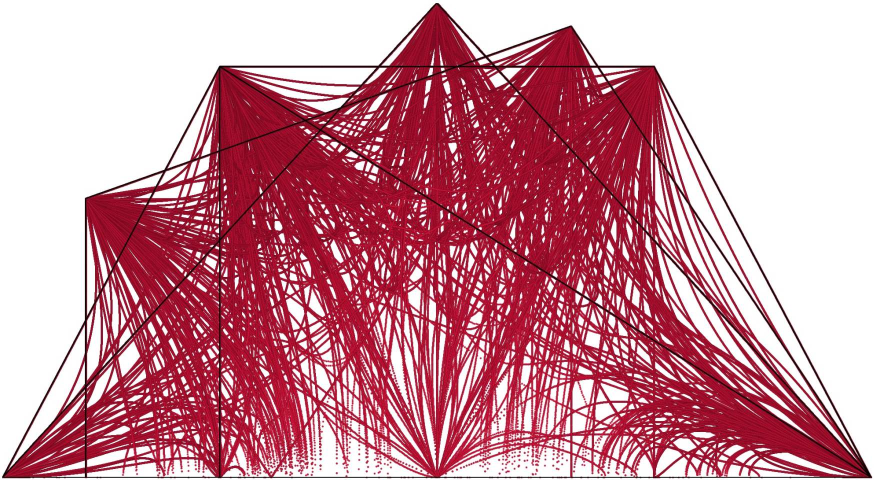
Eigenvalues of convex combinations of pairs of permutations matrices for $n = 5$.

The Numerical Computations of Harlev–Johnson–Lim



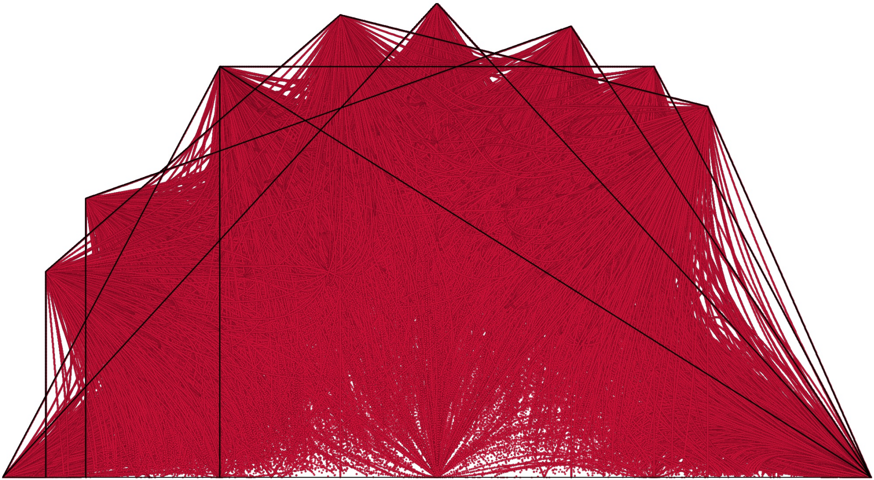
Eigenvalues of convex combinations of triples of permutations matrices for $n = 5$.

The Numerical Computations of Harlev–Johnson–Lim



Eigenvalues of convex combinations of pairs of permutations matrices for $n = 6$.

The Numerical Computations of Harlev–Johnson–Lim



Eigenvalues of convex combinations of pairs of permutations matrices for $n = 7$.