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A matrix generalization of the Hardy-Littlewood-Pólya rearrangement inequality and its applications. (English. English summary)

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The classical Hardy-Littlewood-Pólya rearrangement inequality states that for any vectors $u, v \in \mathbb{R}^n$,

$$\sum_{i=1}^{n} u_i^{\downarrow} v_i^{\uparrow} \le \sum_{i=1}^{n} u_i v_i \le \sum_{i=1}^{n} u_i^{\downarrow} v_i^{\downarrow},$$

where $u^{\downarrow}$ and $v^{\downarrow}$ (resp. $u^{\uparrow}$ and $v^{\uparrow}$) denote the vectors obtained by arranging the entries in decreasing (resp. increasing) order.

The main contribution of the paper is a matrix-valued extension of this classical rearrangement inequality. The author proves that if $f\colon (0,\infty) \to \mathbb{R}$ is a differentiable function such that $s \mapsto sf'(s)$ is monotonically increasing on $(0,\infty)$, then for any positive definite matrices $A, B \in \mathbb{R}^{n\times n}$,

$$\sum_{i=1}^{n} f(\lambda_i(A)\lambda_{n-i+1}(B)) \le \sum_{i=1}^{n} f\Big(\lambda_i(B^{1/2}AB^{1/2})\Big) \le \sum_{i=1}^{n} f(\lambda_i(A)\lambda_i(B)).$$

This result simultaneously generalizes London's convex-function version of the rearrangement inequality and matrix inequalities of Carlen and Lieb.

The proof is based on an optimization problem over the group of orthogonal matrices and a commutation argument showing that, at optimality, the relevant matrices must commute and therefore can be simultaneously diagonalized, modulo a permutation of the diagonal entries.

The paper also derives an analogue for singular values of rectangular matrices.

The second half of the article presents several applications. These include new inequalities for Schatten-q quasi-norms, bounds for the affine-invariant Riemannian metric on the cone of positive definite matrices, and estimates for Alpha-Beta log-determinant divergences arising in information geometry and machine learning.

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.