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Eigenvalue–Eigenvector Identity for Dual Hermitian Matrices

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Abstract. **This paper studies an analogue of the classical eigenvalue–eigenvector identity for dual Hermitian matrices. The classical identity relates the eigenvalues of a Hermitian matrix, the eigenvalues of its principal submatrices, and the entries of its eigenvectors. Assuming that the standard eigenvalues are simple, we establish the corresponding identity for dual Hermitian matrices and show that it naturally decomposes into a standard component and a dual component. The standard component recovers the classical Hermitian identity, while the dual component yields an explicit first-order correction formula relating eigenvalue and eigenvector corrections to perturbations of the corresponding principal submatrices. Numerical examples are presented to verify the theoretical results.**

1 Introduction

For more than a century, the mathematical frameworks of dual numbers and dual quaternions, which we briefly review below, including their corresponding vectors and matrices, have been extensively studied and applied in various fields of mathematics and engineering. The theory of dual numbers originated in the late 19th century [5], and was further developed in the early 20th century through the notion of dual angles and related geometric constructions [33]. These early contributions provided a basis for employing dual numbers, along with their vector and matrix representations, in fields such as kinematics, mechanics, and robotics [1, 11, 13, 19, 20, 27, 28].

In subsequent years [10, 12], dual quaternions, specifically their unit forms, have gained prominence in various modern applications, including neuroscience, hand–eye calibration, cooperative control of multiple agents, and simultaneous localization and mapping (SLAM) [2, 3, 4, 6, 14, 16, 23, 25, 29]. More recently, dual complex matrices [17, 32] have also been utilized in brain signal analysis [30]. These developments illustrate the continuing growth and relevance of dual matrix theory in both theoretical and applied settings.

The eigenvector–eigenvalue identity has evolved through a long and complex historical development. It has appeared, sometimes in slightly different or hidden forms, and has been rediscovered several times in different research areas such as random matrix theory, numerical linear algebra, inverse eigenvalue problems, graph theory (including graph reconstruction, chemical graph theory, and walks on graphs), and even neutrino physics [7, 8, 9, 18, 26, 31].

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While the identity remains valid for all Hermitian matrices and admits extensions to normal and diagonalizable matrices, it is particularly significant in the case of symmetric tridiagonal matrices, such as Jacobi matrices, which play a central role in several core algorithms of numerical linear algebra. In several instances, the identity did not appear explicitly as a direct relation between eigenvalues and eigenvectors, but rather emerged implicitly as an auxiliary result in other analytical developments. The form and notation of the identity also varied widely, making it difficult to locate with normal literature searches.

Interest in the identity was revived following an expository publication by Wolchover in Quanta Magazine [31], which discussed its modern rediscovery by Denton et al. [8, 9]. As reported in that article, Terence Tao remarked that "something this short and simple should have been in textbooks already." This observation reflects the surprising historical trajectory of an identity that had largely escaped attention despite its elementary nature. In subsequent work, Denton et al. [8] presented a concise survey of the identity, including several proofs and generalizations. Their rediscovery arose in the study of numerically stable formulas for the eigenvectors of neutrino oscillation Hamiltonians, where the authors derived the identity explicitly in the case of 3×3 matrices [9].

The study of eigenvalue problems for dual matrices remains in its early stages and continues to offer opportunities for further investigation. Pennestrì and Stefanelli [19, 20] formulated the problem of determining the eigenvalues and the singular values of matrices defined over dual numbers. Qi and Luo [24] demonstrated that a dual quaternion Hermitian matrix of order $n \times n$ possesses n dual number eigenvalues and further established the singular value decomposition (SVD) for dual quaternion matrices. These results are also applicable to dual number matrices and dual complex matrices [22].

More recently, Qi and Cui [21] introduced a supplementary matrix approach for computing the eigenvalues of dual Hermitian matrices and demonstrated its applicability to problems in multi-agent formation control. Motivated by these developments, we establish in the present work an eigenvalue–eigenvector identity for dual Hermitian matrices.

To the best of our knowledge, no analogue of the eigenvalue–eigenvector identity has previously been established for dual Hermitian matrices. The primary contribution of this paper is the derivation of an explicit first-order correction to the classical eigenvalue–eigenvector identity in the setting of dual Hermitian matrices. Specifically, we show that the dual eigenvalue–eigenvector identity naturally decomposes into two parts. The standard component recovers the classical identity for Hermitian matrices, while the dual component yields an explicit first-order correction formula relating the corrections to the eigenvalues and eigenvectors to the perturbations of the corresponding principal submatrices. This decomposition extends the classical identity to the dual setting and provides a clear interpretation of the effect of infinitesimal perturbations on the spectral structure of dual Hermitian matrices.

To illustrate the proposed identities, we present two examples and verify the results using the supplementary matrix framework introduced by Qi and Cui [21] for computing the eigenvalues and eigenvectors of dual Hermitian matrices.

2 Preliminaries

Let $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers, respectively, and let $\mathbb{K}$ represent either $\mathbb{R}$ or $\mathbb{C}$. The corresponding sets of dual numbers and dual complex numbers are denoted by $\widehat{\mathbb{R}}$ and $\widehat{\mathbb{C}}$, respectively. We write $\widehat{\mathbb{K}}$ to denote either of these two sets. An element of $\widehat{\mathbb{K}}$ will be called a *dual element*.

A dual element is an expression of the form $a = a_s + a_d\varepsilon$, where $a_s, a_d \in \mathbb{K}$, and ε is an infinitesimal unit satisfying $\varepsilon^2 = 0$ and commuting with all real or complex scalars. The quantities a_s and a_d are called the standard part and dual part of a, respectively. The conjugate of a is defined by $a^* = a_s^* + a_d^*\varepsilon$, where a_s^* and a_d^* denote the ordinary conjugates of a_s and a_d. A dual element is said to be *appreciable* if $a_s \neq 0$, and *infinitesimal* otherwise. Moreover, a dual element is said to be *invertible* if there exists a dual element b such that $ab = ba = 1$. Equivalently, a dual element is invertible if and only if it is appreciable (see, [21]).

A dual matrix $A = A_s + \varepsilon A_d \in \widehat{\mathbb{K}}^{n\times n}$ is said to be *Hermitian* if $A = A^*$. A dual vector $x = x_s + x_d\varepsilon$, where $x_s,\ x_d \in \mathbb{K}^n$, is called a *unit dual vector* if $\|x_s\|_2 = 1$ and $x_s^* x_d + x_d^* x_s = 0$, where $\|\cdot\|_2$ denotes the standard Euclidean norm.

Let $A = A_s + \varepsilon A_d \in \widehat{\mathbb{K}}^{n\times n}$ be a dual Hermitian matrix. Suppose that $Av_i = \lambda_i v_i$, where $\lambda_i = \lambda_{i,s} + \varepsilon\lambda_{i,d}$ is the i-th eigenvalue of A, and $v_i = u_i + \varepsilon w_i$ is a corresponding unit dual eigenvector, for $i = 1,\dots,n$. Then, $\|u_i\|_2 = 1$ and $u_i^* w_i + w_i^* u_i = 0$, for $i = 1,\dots,n$. Throughout the manuscript, $\mathrm{Re}(z)$ denotes the real part of a complex number $z \in \mathbb{C}$.

Let $p_A : \mathbb{C} \longrightarrow \mathbb{C}$ denote the characteristic polynomial of A. Then

$$p_A(\lambda) = \det(\lambda I_n - A) = \prod_{k=1}^{n}(\lambda - \lambda_k),$$

where I_n (or I) represents the $n \times n$ identity matrix.

Evaluating the derivative of p_A at $\lambda = \lambda_i$ gives

$$P_i := p_A'(\lambda_i) = \prod_{k\neq i}(\lambda_i - \lambda_k).$$

For $j = 1,\dots,n$, let M_j denote the $(n-1)\times(n-1)$ principal submatrix of A obtained by deleting the j-th row and column. Its characteristic polynomial is

$$p_{M_j}(\lambda) = \det(\lambda I_{n-1} - M_j) = \prod_{k=1}^{n-1}\big(\lambda - \lambda_k(M_j)\big).$$

The following result of Qi and Cui [21] provides the necessary framework for the subsequent results. Theorem 4.1 of [21] states that for a given dual Hermitian matrix $A = A_s + \varepsilon A_d \in \widehat{\mathbb{K}}^{n\times n}$, let $\lambda_{i,s}$ be a simple eigenvalue of A_s, and u_i be the corresponding unit eigenvector of $\lambda_{i,s}$ for $i = 1,\dots,n$. Then, $\lambda_i = \lambda_{i,s} + \varepsilon\lambda_{i,d}$ is a simple eigenvalue of A, where $\lambda_{i,d} = u_i^* A_d u_i$, with the corresponding eigenvector $v_i = u_i + \varepsilon w_i$, where w_i satisfies

$$(A_s - \lambda_{i,s} I)w_i = (\lambda_{i,d} I - A_d)u_i,$$

for $i = 1,\dots,n$.

3 Main results

The following proposition establishes the dual eigenvalue–eigenvector identity for dual Hermitian matrices with simple standard eigenvalues. Its proof adapts the adjugate-matrix argument of Denton et al. [8] to the dual setting.

Proposition 3.1. *Let $A = A_s + \varepsilon A_d \in \widehat{\mathbb{K}}^{n\times n}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, be a dual Hermitian matrix. Assume that the eigenvalues $\lambda_{1,s}, \ldots, \lambda_{n,s}$ of A_s are simple. Let*

$$Av_i = \lambda_i v_i, \quad \lambda_i = \lambda_{i,s} + \varepsilon\lambda_{i,d},$$

where v_i is a unit dual eigenvector corresponding to λ_i. Let M_j be the principal submatrix of A obtained by deleting the j-th row and column. Then, for each $i, j = 1, \ldots, n$,

$$p_{M_j}(\lambda_i) = |v_{i,j}|^2\, p'_A(\lambda_i), \tag{3.1}$$

where $v_{i,j}$ is the j-th component of the unit dual eigenvector v_i associated with λ_i.

Proof By Theorem 4.1 of Qi and Cui [21], for each simple eigenvalue $\lambda_{i,s}$ of A_s, there exists a dual eigenvalue

$$\lambda_i = \lambda_{i,s} + \varepsilon\lambda_{i,d}$$

and an associated appreciable dual eigenvector $v_i = u_i + \varepsilon w_i$. The eigenvectors may be normalized so that $v_i^* v_i = 1$.

We first prove that $v_1, \ldots, v_n$ are mutually orthogonal. Since $\lambda_k = \lambda_{k,s} + \varepsilon\lambda_{k,d}$, we have

$$\lambda_k^* = \lambda_{k,s}^* + \varepsilon\lambda_{k,d}^*.$$

Because A_s is Hermitian, $\lambda_{k,s} \in \mathbb{R}$. Moreover,

$$\lambda_{k,d} = u_k^* A_d u_k.$$

Since A_d is Hermitian, $\lambda_{k,d} \in \mathbb{R}$. Hence

$$\lambda_k^* = \lambda_k.$$

Since A is dual Hermitian, $A^* = A$. For $i \neq k$, we have

$$Av_i = \lambda_i v_i, \qquad Av_k = \lambda_k v_k.$$

Thus,

$$v_k^* A v_i = \lambda_i v_k^* v_i.$$

On the other hand,

$$v_k^* A v_i = (Av_k)^* v_i = (\lambda_k v_k)^* v_i = \lambda_k^* v_k^* v_i = \lambda_k v_k^* v_i.$$

It follows that

$$(\lambda_i - \lambda_k) v_k^* v_i = 0.$$

Since $\lambda_{i,s} \neq \lambda_{k,s}$, the dual number

$$\lambda_i - \lambda_k = (\lambda_{i,s} - \lambda_{k,s}) + \varepsilon(\lambda_{i,d} - \lambda_{k,d})$$

has nonzero standard part and is therefore invertible. Hence

$$v_k^* v_i = 0 \qquad (i \neq k).$$

Together with $v_i^* v_i = 1$, this gives

$$V^* V = I, \qquad V = [v_1, \dots, v_n].$$

Thus V is dual unitary. Indeed, $V^* V = I$ implies

$$\det(V)^* \det(V) = 1,$$

so $\det(V)$ is invertible in the dual-number algebra. Hence V is invertible and $V^{-1} = V^*$.

Since $A v_i = \lambda_i v_i$ for all i, we have

$$AV = V\Lambda, \qquad \Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n).$$

Multiplying on the right by V^*, we obtain

$$A = V\Lambda V^*.$$

Now let λ be a dual number. Then

$$\lambda I - A = V(\lambda I - \Lambda)V^*.$$

The determinant and adjugate are well defined over the commutative algebra of dual numbers. Since the entries belong to a commutative algebra, the standard polynomial identity

$$\operatorname{adj}(PXP^{-1}) = P \operatorname{adj}(X) P^{-1}$$

holds for every invertible matrix P. This follows, for instance, from

$$\operatorname{adj}(AB) = \operatorname{adj}(B) \operatorname{adj}(A),$$

which is a polynomial identity valid over every commutative ring. Therefore,

$$\operatorname{adj}(\lambda I - A) = V \operatorname{adj}(\lambda I - \Lambda) V^*.$$

Since $\lambda I - \Lambda$ is diagonal, we have

$$\operatorname{adj}(\lambda I - \Lambda) = \operatorname{diag}\left(\prod_{m \neq 1} (\lambda - \lambda_m), \dots, \prod_{m \neq n} (\lambda - \lambda_m) \right).$$

Putting $\lambda = \lambda_i$, all diagonal entries of $\operatorname{adj}(\lambda_i I - \Lambda)$ vanish except the i-th one. Hence

$$\operatorname{adj}(\lambda_i I - A) = \left(\prod_{m \neq i} (\lambda_i - \lambda_m) \right) v_i v_i^*.$$

Taking the (j, j)-entry of both sides gives

$$\det(\lambda_i I_{n-1} - M_j) = \left(\prod_{m \neq i} (\lambda_i - \lambda_m) \right) v_{i,j} v_{i,j}^*.$$

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Since $v_{i,j}v_{i,j}^* = |v_{i,j}|^2$, we get

$$p_{M_j}(\lambda_i) = |v_{i,j}|^2 \prod_{m\neq i}(\lambda_i - \lambda_m).$$

It remains to identify the product with $p'_A(\lambda_i)$. Since $A = V\Lambda V^*$ and $V^* = V^{-1}$, we have

$$\begin{aligned} p_A(\lambda) &= \det(\lambda I - A) = \det\left(V(\lambda I - \Lambda)V^*\right) \\ &= \det(V)\det(\lambda I - \Lambda)\det(V^*) = \det(\lambda I - \Lambda) = \prod_{m=1}^{n}(\lambda - \lambda_m). \end{aligned}$$

Therefore,

$$p'_A(\lambda_i) = \prod_{m\neq i}(\lambda_i - \lambda_m).$$

Hence

$$p_{M_j}(\lambda_i) = |v_{i,j}|^2 p'_A(\lambda_i).$$

This completes the proof. ■

Assume that the standard eigenvalues $\lambda_{1,s}, \ldots, \lambda_{n,s}$ of A_s are simple. The dual Hermitian principal submatrix M_j of order $(n-1)\times(n-1)$, obtained by removing the j-th row and column of A, can be written in standard and dual parts as $M_j = M_{j,s} + \varepsilon M_{j,d}$ for $j = 1, \ldots, n$. For $i = 1, \ldots, n$, denote

$$p_{M_{j,s}}(\lambda_{i,s}) = \det(\lambda_{i,s}I - M_{j,s}),$$

$$p'_{M_{j,s}}(\lambda_{i,s}) = \frac{d}{d\lambda}(\det(\lambda I - M_{j,s}))\big|_{\lambda=\lambda_{i,s}},$$

$$p'_{A_s}(\lambda_{i,s}) = \prod_{k\neq i}(\lambda_{i,s} - \lambda_{k,s}) := P_{s,i}.$$

By the Jacobi formula [15], we obtain

$$\operatorname{tr}(\operatorname{adj}(\lambda_{i,s}I - M_{j,s})) = \frac{d}{d\lambda}(\det(\lambda I - M_{j,s}))\big|_{\lambda=\lambda_{i,s}} = p'_{M_{j,s}}(\lambda_{i,s}). \tag{3.2}$$

The following theorem decomposes the dual eigenvalue–eigenvector identity into its standard and dual components. The standard component coincides with the classical identity, whereas the dual component provides an explicit first-order correction formula describing the interaction between the perturbations of the eigenvalues, eigenvectors, and principal submatrices.

***Theorem* 3.2** *Let $A = A_s + \varepsilon A_d \in \widehat{\mathbb{K}}^{n\times n}$ be a dual Hermitian matrix. Assume that the standard eigenvalues $\lambda_{1,s}, \ldots, \lambda_{n,s}$ of A_s are simple. Let $v_i = u_i + \varepsilon w_i$ be the unit dual vector corresponding to $\lambda_i = \lambda_{i,s} + \varepsilon\lambda_{i,d}$, $i = 1, \ldots, n$. Then, for each $j = 1, \ldots, n$, we have*

(1) Standard identity:

$$|u_{i,j}|^2\, p'_{A_s}(\lambda_{i,s}) = p_{M_{j,s}}(\lambda_{i,s}) \tag{3.3}$$

(2) Dual correction identity:

$$2\,\mathrm{Re}\left(u_{i,j}\overline{w_{i,j}}\right)\, p'_{A_s}(\lambda_{i,s}) + |u_{i,j}|^2\, p'_{A_s}(\lambda_{i,s}) \sum_{m\neq i} \frac{\lambda_{i,d}-\lambda_{m,d}}{\lambda_{i,s}-\lambda_{m,s}} = \lambda_{i,d}\, p'_{M_{j,s}}(\lambda_{i,s}) - \mathrm{tr}\big(\mathrm{adj}(\lambda_{i,s}I - M_{j,s})M_{j,d}\big)\,, \tag{3.4}$$

where $u_{i,j}$ and $w_{i,j}$ denote the j-th components of u_i and w_i, respectively.

Proof Let $X_s := \lambda_{i,s}I - M_{j,s}$ and $Y := \lambda_{i,d}I - M_{j,d}$. Then

$$\begin{aligned} p_{M_j}(\lambda_i) &= \det(\lambda_i I - M_j) \\ &= \det\big((\lambda_{i,s} + \varepsilon\lambda_{i,d})I - (M_{j,s} + \varepsilon M_{j,d})\big) \\ &= \det(X_s + \varepsilon Y) \end{aligned}$$

From Jacobi's formula [15], which states that for any differentiable matrix function $X(t)$,

$$\frac{d}{dt}\det(X(t)) = \mathrm{tr}\left(\mathrm{adj}(X(t))\frac{dX(t)}{dt}\right).$$

Consider

$$X(t) = X_s + tY.$$

Then the scalar function $f(t) := \det(X(t))$ has the first-order Taylor expansion about $t = 0$,

$$f(t) = f(0) + t f'(0) + O(t^2).$$

Since $f(0) = \det(X_s)$ and, by Jacobi's formula,

$$f'(0) = \mathrm{tr}(\mathrm{adj}(X_s)Y)\,,$$

we obtain

$$\det(X_s + tY) = \det(X_s) + t\,\mathrm{tr}(\mathrm{adj}(X_s)Y) + O(t^2).$$

Substituting $t = \varepsilon$ and using $\varepsilon^2 = 0$ yields

$$\det(X_s + \varepsilon Y) = \det(X_s) + \varepsilon\,\mathrm{tr}(\mathrm{adj}(X_s)Y)\,.$$

Now

$$\det(X_s) = \det(\lambda_{i,s}I - M_{j,s}) = p_{M_{j,s}}(\lambda_{i,s}),$$

and

$$\begin{aligned} \mathrm{tr}\left(\mathrm{adj}(X_s)Y\right) &= \mathrm{tr}\left(\mathrm{adj}(\lambda_{i,s}I - M_{j,s})(\lambda_{i,d}I - M_{j,d})\right) \\ &= \lambda_{i,d}\,\mathrm{tr}\left(\mathrm{adj}(\lambda_{i,s}I - M_{j,s})\right) - \mathrm{tr}\left(\mathrm{adj}(\lambda_{i,s}I - M_{j,s})M_{j,d}\right). \end{aligned}$$

By (3.2),

$$\mathrm{tr}\left(\mathrm{adj}(\lambda_{i,s}I - M_{j,s})\right) = \left.\frac{d}{d\lambda}\det(\lambda I - M_{j,s})\right|_{\lambda=\lambda_{i,s}} = p'_{M_{j,s}}(\lambda_{i,s})\,.$$

Hence

$$p_{M_j}(\lambda_i) = p_{M_{j,s}}(\lambda_{i,s}) + \varepsilon\left(\lambda_{i,d}p'_{M_{j,s}}(\lambda_{i,s}) - \mathrm{tr}\left(\mathrm{adj}(\lambda_{i,s}I - M_{j,s})M_{j,d}\right)\right). \tag{3.5}$$

Now

$$P_i = \prod_{k\neq i}(\lambda_i - \lambda_k) = \prod_{k\neq i}\big((\lambda_{i,s} - \lambda_{k,s}) + \varepsilon(\lambda_{i,d} - \lambda_{k,d})\big).$$

Since the standard eigenvalues are simple, we have

$$\lambda_{i,s} - \lambda_{k,s} \neq 0, \quad k \neq i.$$

Hence each dual number

$$(\lambda_{i,s} - \lambda_{k,s}) + \varepsilon(\lambda_{i,d} - \lambda_{k,d})$$

has a nonzero standard part and is therefore invertible in the dual-number algebra. Factoring out the standard part from each factor, we obtain

$$P_i = \prod_{k\neq i}(\lambda_{i,s} - \lambda_{k,s})\prod_{k\neq i}\left(1 + \varepsilon\frac{\lambda_{i,d} - \lambda_{k,d}}{\lambda_{i,s} - \lambda_{k,s}}\right).$$

Setting

$$P_{s,i} = \prod_{k\neq i}(\lambda_{i,s} - \lambda_{k,s}).$$

Since $\varepsilon^2 = 0$, every term in the product containing two or more factors of ε vanishes. Consequently, only the constant term and the terms containing exactly one factor of ε contribute to the expansion. Therefore,

$$\prod_{k\neq i}\left(1 + \varepsilon\frac{\lambda_{i,d} - \lambda_{k,d}}{\lambda_{i,s} - \lambda_{k,s}}\right) = 1 + \varepsilon\sum_{k\neq i}\frac{\lambda_{i,d} - \lambda_{k,d}}{\lambda_{i,s} - \lambda_{k,s}}.$$

Hence

$$P_i = P_{s,i} + \varepsilon P_{s,i}\sum_{k\neq i}\frac{\lambda_{i,d} - \lambda_{k,d}}{\lambda_{i,s} - \lambda_{k,s}}.$$

Setting

$$P_{d,i} := P_{s,i}\sum_{k\neq i}\frac{\lambda_{i,d} - \lambda_{k,d}}{\lambda_{i,s} - \lambda_{k,s}},$$

we obtain

$$P_i = P_{s,i} + \varepsilon P_{d,i}. \tag{3.6}$$

Now we expand $|v_{i,j}|^2$ as

$$\begin{aligned}|v_{i,j}|^2 &= (u_{i,j} + \varepsilon w_{i,j})(\overline{u_{i,j}} + \varepsilon\overline{w_{i,j}})\\ &= u_{i,j}\overline{u_{i,j}} + \varepsilon(u_{i,j}\overline{w_{i,j}} + w_{i,j}\overline{u_{i,j}}) + \varepsilon^2(w_{i,j}\overline{w_{i,j}}).\end{aligned}$$

Since $\varepsilon^2 = 0$, the last term vanishes. Therefore,

$$|v_{i,j}|^2 = u_{i,j}\overline{u_{i,j}} + \varepsilon(u_{i,j}\overline{w_{i,j}} + w_{i,j}\overline{u_{i,j}}) = |u_{i,j}|^2 + 2\varepsilon\,\mathrm{Re}(u_{i,j}\overline{w_{i,j}}). \tag{3.7}$$

Using (3.6) and (3.7), we obtain

$$\begin{aligned}|v_{i,j}|^2\, p_A'(\lambda_i) &= (P_{s,i} + \varepsilon P_{d,i})(|u_{i,j}|^2 + 2\varepsilon\,\mathrm{Re}(u_{i,j}\overline{w_{i,j}}))\\ &= P_{s,i}|u_{i,j}|^2 + \varepsilon\big(2\,\mathrm{Re}(u_{i,j}\overline{w_{i,j}})P_{s,i} + |u_{i,j}|^2 P_{d,i}\big).\end{aligned} \tag{3.8}$$

Using (3.5) and (3.8) in (3.1), we obtain

$$\begin{aligned} p_{M_{j,s}}(\lambda_{i,s}) + \varepsilon\Big(\lambda_{i,d}\, p'_{M_{j,s}}(\lambda_{i,s}) - \operatorname{tr}\big(\operatorname{adj}(\lambda_{i,s}I - M_{j,s})M_{j,d}\big)\Big) \\ = P_{s,i}|u_{i,j}|^2 + \varepsilon\Big(2\operatorname{Re}\left(u_{i,j}\overline{w_{i,j}}\right) P_{s,i} + |u_{i,j}|^2 P_{d,i}\Big). \end{aligned} \tag{3.9}$$

Equating the standard (non ε) parts of (3.9) gives the standard identity (3.3). Substituting the value $P_{s,i}$ and $P_{d,i}$, and equating the ε parts of (3.9) gives the dual-correction identity (3.4). ■

Corollary 3.3. *Under the assumptions of Theorem 3.2, identity (3.1) holds if and only if the standard identity (3.3) and the dual correction identity (3.4) are simultaneously satisfied.*

Proof From the proof of Theorem 3.2, equation (3.9) is precisely the decomposition of

$$p_{M_j}(\lambda_i) = |v_{i,j}|^2\, p'_A(\lambda_i)$$

into its standard and dual parts. By equating the standard and dual parts of both sides, we obtain (3.3) and (3.4). Conversely, if (3.3) and (3.4) hold, then the standard and dual parts of both sides coincide, and therefore identity (3.1) holds. ■

Remark 3.4 Equation (3.4) is new and couples the eigenvalue corrections $\lambda_{i,d}$, the eigenvector corrections w_i, and the dual perturbations $A_d, M_{j,d}$. It provides an explicit first-order correction to the classical eigenvalue–eigenvector identity. It quantifies how perturbations in the dual part of a Hermitian matrix influence both the eigenvalue corrections and the eigenvector corrections through the corresponding principal submatrices.

4 Illustrative examples

The following examples illustrate the main ideas of Theorem 3.2 in both symbolic and numerical settings. We first consider a symbolic 2×2 example to highlight the structure of the standard and dual correction identities in a simple setting. We then present a numerical 3×3 example to demonstrate how the identities can be verified concretely for nontrivial dual Hermitian matrices.

Example 4.1 Consider the dual Hermitian matrix

$$A = \begin{pmatrix} a & c \\ \overline{c} & d \end{pmatrix} + \varepsilon \begin{pmatrix} \alpha & \gamma \\ \overline{\gamma} & \delta \end{pmatrix},$$

where $a, d, \alpha, \delta \in \mathbb{R}$ and $c, \gamma \in \mathbb{C}$.
Assume that the standard part A_s has distinct eigenvalues

$$\lambda_{1,s},\ \lambda_{2,s},$$

with corresponding orthonormal eigenvectors

$$u_1 = \begin{pmatrix} u_{1,1} \\ u_{1,2} \end{pmatrix}, \text{ and } u_2 = \begin{pmatrix} u_{2,1} \\ u_{2,2} \end{pmatrix}.$$

Let

$$\lambda_i = \lambda_{i,s} + \varepsilon\lambda_{i,d}, \text{ and } v_i = u_i + \varepsilon w_i,$$

for $i = 1, 2$. For $i = j = 1$, the principal submatrix is

$$M_1 = d + \varepsilon\delta,$$

so that

$$p_{M_1}(\lambda_1) = (\lambda_{1,s} - d) + \varepsilon(\lambda_{1,d} - \delta).$$

Since

$$p'_A(\lambda_1) = \lambda_1 - \lambda_2,$$

we have

$$p'_A(\lambda_1) = (\lambda_{1,s} - \lambda_{2,s}) + \varepsilon(\lambda_{1,d} - \lambda_{2,d}).$$

Moreover,

$$|v_{1,1}|^2 = |u_{1,1}|^2 + 2\varepsilon\,\mathrm{Re}(u_{1,1}\overline{w_{1,1}}).$$

Using

$$p_{M_1}(\lambda_1) = |v_{1,1}|^2 p'_A(\lambda_1),$$

and comparing standard and dual parts, we obtain

$$\lambda_{1,s} - d = |u_{1,1}|^2(\lambda_{1,s} - \lambda_{2,s}),$$

which is the classical eigenvector–eigenvalue identity, together with the dual correction identity

$$\lambda_{1,d} - \delta = 2\,\mathrm{Re}(u_{1,1}\overline{w_{1,1}})(\lambda_{1,s} - \lambda_{2,s}) + |u_{1,1}|^2(\lambda_{1,d} - \lambda_{2,d}).$$

This example illustrates how the dual identity naturally extends the classical relation through an additional correction term.

Example 4.2 Consider a dual Hermitian matrix $A = A_s + \varepsilon A_d \in \widehat{\mathbb{R}}^{3\times 3}$ with A_s and A_d given as

$$A_s = \begin{pmatrix} 2 & 1 & 0.5 \\ 1 & 3 & 0.2 \\ 0.5 & 0.2 & 1 \end{pmatrix} \quad \text{and} \quad A_d = \begin{pmatrix} 0.10 & 0.50 & -0.20 \\ 0.50 & 0 & 0.3 \\ -0.20 & 0.3 & -0.05 \end{pmatrix} \tag{4.1}$$

For each simple standard eigenvalue $\lambda_{i,s}$, $i = 1, 2, 3$, a unit eigenvector u_i of A_s is computed. The corresponding dual eigenvalue correction is given by

$$\lambda_{i,d} = u_i^* A_d u_i.$$

The eigenvector correction w_i is then obtained by solving

$$(A_s - \lambda_{i,s} I)w_i = (\lambda_{i,d} I - A_d)u_i,$$

as given by Theorem 4.1 of Qi and Cui [21]. The corresponding dual eigenvectors are

$$v_i = u_i + \varepsilon w_i, \quad i = 1, 2, 3,$$

where v_i is chosen so that $v_i^* v_i = 1$. We now verify Theorem 3.2 for the dominant

i	Eigenvalue $\lambda_i = \lambda_{i,s} + \varepsilon\lambda_{i,d}$	u_i	w_i
1	$3.68893048 + \varepsilon\, 0.51805479$	$\begin{bmatrix} -0.53776387 \\ -0.82747448 \\ -0.16154260 \end{bmatrix}$	$\begin{bmatrix} -0.07054694 \\ 0.05121676 \\ -0.02750337 \end{bmatrix}$
2	$1.54365251 - \varepsilon\, 0.62078441$	$\begin{bmatrix} -0.70673578 \\ 0.54691183 \\ -0.44878947 \end{bmatrix}$	$\begin{bmatrix} -0.00574458 \\ 0.08965822 \\ 0.11830724 \end{bmatrix}$
3	$0.76741701 + \varepsilon\, 0.15272962$	$\begin{bmatrix} -0.45971139 \\ 0.12717482 \\ 0.87891524 \end{bmatrix}$	$\begin{bmatrix} 0.09135623 \\ -0.05232624 \\ 0.05535469 \end{bmatrix}$

Table 1: Computation of eigenvalues and eigenvectors of $A = A_s + \varepsilon A_d$, where A_s and A_d are given in (4.1)

eigenvalue

$$\lambda_1 = 3.68893048 + \varepsilon\, 0.51805479.$$

The purpose of this example is to illustrate explicitly how the standard identity and the dual correction identity interact with the principal submatrices of A.

We first consider the case $j = 1$, corresponding to the removal of the first row and column of A. In this case, the quantities appearing in equation (3.4) are computed directly from the associated principal submatrices $M_{1,s}$ and $M_{1,d}$.
Corresponding to $i = j = 1$, the matrices $M_{1,s}$ and $M_{1,d}$ are obtained as

$$M_{1,s} = \begin{pmatrix} 3 & 0.2 \\ 0.2 & 1 \end{pmatrix} \text{ and } M_{1,d} = \begin{pmatrix} 0 & 0.3 \\ 0.3 & -0.05 \end{pmatrix},$$

respectively. For the dual correction identity (3.4),

$$\operatorname{adj}(\lambda_{1,s} I - M_{1,s}) = \begin{pmatrix} \lambda_{1,s} - 1 & 0.2 \\ 0.2 & \lambda_{1,s} - 3 \end{pmatrix}.$$

Using

$$p'_{M_{1,s}}(\lambda_{1,s}) = \operatorname{tr}(\operatorname{adj}(\lambda_{1,s} I - M_{1,s})),$$

we obtain

$$p'_{M_{1,s}}(\lambda_{1,s}) = (\lambda_{1,s} - 1) + (\lambda_{1,s} - 3) = 3.377861.$$

Moreover,

$$\operatorname{tr}\left(\operatorname{adj}(\lambda_{1,s} I - M_{1,s}) M_{1,d}\right) = \frac{3}{25} - \frac{\lambda_{1,s} - 3}{20} = 0.085553.$$

- The quantity

$$P_{s,1} = \prod_{k\neq 1}(\lambda_{1,s} - \lambda_{k,s}) = p'_{A_s}(\lambda_{1,s})$$

measures the spectral separation of the eigenvalue $\lambda_{1,s}$ from the remaining eigenvalues of A_s. Numerically,

$$P_{s,1} = 6.267458.$$

- The corresponding dual correction term is

$$P_{d,1} = P_{s,1}\sum_{k\neq 1}\frac{\lambda_{1,d} - \lambda_{k,d}}{\lambda_{1,s} - \lambda_{k,s}},$$

which captures the perturbative contribution of the dual eigenvalue corrections. In this example,

$$P_{d,1} = 4.110858.$$

- The left-hand side of the dual correction identity (3.4) is therefore

$$\mathrm{LHS}_\varepsilon(1,1) := \lambda_{1,d}p'_{M_{1,s}}(\lambda_{1,s}) - \mathrm{tr}(\mathrm{adj}(\lambda_{1,s}I - M_{1,s})M_{1,d}) = 1.664364.$$

- The right-hand side becomes

$$\mathrm{RHS}_\varepsilon(1,1) := 2\,\mathrm{Re}(u_{1,1}\overline{w_{1,1}})P_{s,1} + |u_{1,1}|^2P_{d,1} = 1.664364.$$

Since $\mathrm{LHS}_\varepsilon(1,1) = \mathrm{RHS}_\varepsilon(1,1) = 1.664364$, the dual identity is verified for $i = j = 1$.

For the standard identity (3.3), we compute

$$p_{M_{1,s}}(\lambda_{1,s}) = \det(\lambda_{1,s}I - M_{1,s}) = 1.812486,$$

while

$$P_{s,1}|u_{1,1}|^2 = 1.812486.$$

Thus the standard identity is also verified. Since $p_{M_{1,s}}(\lambda_{1,s}) = P_{s,1}|u_{1,1}|^2 = 1.812486$, the standard identity is verified for $i = j = 1$.

The same procedure may be repeated for $j = 2$. In this case,

$$M_{2,s} = \begin{pmatrix} 2 & 0.5 \\ 0.5 & 1 \end{pmatrix}, \text{ and } M_{2,d} = \begin{pmatrix} 0.10 & -0.20 \\ -0.20 & -0.05 \end{pmatrix}.$$

Substituting these matrices into (3.4) gives

$$\mathrm{LHS}_\varepsilon(1,2) = 2.283519,$$

and

$$\mathrm{RHS}_\varepsilon(1,2) = 2.283525,$$

which agree up to rounding error, thereby confirming the dual correction identity. For standard identity (3.3),

- $p_{M_{2,s}}(\lambda_{1,s}) = \det(\lambda_{1,s}I - M_{2,s}) = (\lambda_{1,s} - 3)(\lambda_{1,s} - 1) - 0.04 = 1.812486$
- $P_{s,1}|u_{1,2}|^2 = 1.812486$

Since $p_{M_{2,s}}(\lambda_{1,s}) = P_{s,1}|u_{1,2}|^2 = 1.812486$, the standard identity is verified for $i = 1$, $j = 2$. Similarly, the results can be verified for other values of $i, j = 1, 2, 3$.

This example shows how the classical eigenvector–eigenvalue identity for the standard part A_s extends to the dual setting through an additional correction term involving both the dual perturbation matrix and the dual eigenvector corrections.

5 Conclusion

In this work, we established an eigenvalue–eigenvector identity for dual Hermitian matrices. The classical identity was decomposed into standard and dual components, leading to an explicit dual correction formula involving eigenvalue perturbations, eigenvector corrections, and principal submatrices.

Several directions remain open for future investigation. These include extensions to dual normal matrices, dual quaternion Hermitian matrices, and more general structured eigenvalue problems. It would also be of interest to investigate whether analogous correction identities arise in higher-order nilpotent or perturbative algebraic settings.

Conflict of Interest

The authors declare that they have no possible conflicts of interest.

Data availability

No datasets generated or analyzed during this work.

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