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Properties of best approximations with respect to the Ky Fan p-k norm, and the strict spectral approximant of a matrix. (English) Zbl 08213463

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Let $M_{m\times n}(\mathbb{C})$ be endowed with a given norm $\|\cdot\|$, let $\mathcal{M}$ be a linear subspace, and let $A\in M_{m\times n}(\mathbb{C})$. A *best approximation* of A is a matrix $Y\in\mathcal{M}$ minimizing $\|A-Y\|$. For the Schatten p-norms with $1<p<\infty$, strict convexity implies that the best approximation is unique. The situation is different for the spectral norm, for which best approximations need not be unique. To distinguish a canonical choice, Ziętak introduced the notion of a *strict spectral approximation*: among all spectral approximations, one selects the matrix whose error has lexicographically minimal singular-value vector. The Ky Fan p-k norms, defined as the ℓ_p-norm of the k largest singular values of a matrix, provide a natural framework for studying this question. A central question is whether the corresponding best approximations converge to the strict spectral approximation as $p\to\infty$.

The main contribution of the paper is an explicit description of the subdifferential of the Ky Fan p-k norms for $2\leq p<\infty$. Using this description, the authors obtain characterizations of best approximations, Birkhoff-James orthogonality, ε-Birkhoff orthogonality, norm parallelism, and orthogonality to linear subspaces with respect to these norms. The authors also prove uniqueness of best approximations to one-dimensional subspaces under suitable rank assumptions and obtain partial results on the convergence of Ky Fan p-k approximations to the strict spectral approximation, thereby answering several questions raised by Ziętak. The arguments combine techniques from convex analysis, subdifferential calculus, Fréchet differentiation of matrix functions, and the theory of unitarily invariant norms.

Reviewer: Frédéric Morneau-Guérin (Québec)

MSC:

15A60 Norms of matrices, numerical range, applications of functional analysis to matrix theory
41A50 Best approximation, Chebyshev systems
41A52 Uniqueness of best approximation
46B20 Geometry and structure of normed linear spaces

Keywords:

Ky Fan p-k norms; subdifferential of norm; best approximations of matrices; approximate orthogonality; Pólya algorithm; strict spectral approximant

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