Between Addition, Multiplication and Beyond: Hyperoperations of Non-integral Ranks

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Peano Arithmetic

Definition (Addition)

The addition function $+ : \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$, is defined inductively by:

1. $a + 0 = a$, and
2. $a + s(b) = s(a + b)$.

Definition (Multiplication)

The multiplication function $\times : \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$, is defined inductively by:

1. $a \times 0 = 0$, and
2. $a \times s(b) = a + (a \times b)$.

Giuseppe Peano

Peano Arithmetic

Definition (Exponentiation)

The addition function $\wedge : \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$, is defined inductively by:

1. $a \wedge 0 = 1$, and
2. $a \wedge s(b) = a \times (a \wedge b)$.

Arithmetic operation is defined *inductively* by

1. a "base case", and
2. a recursion using "previous operation".

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Hyperoperation

Definition (Hyperoperation Hierarchy)

The hyperoperation of rank $n \in \mathbb{Z}_{\geq 0}$, denoted by $H_n : \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$, is defined inductively by:

$$H_n(a, b) = \begin{cases} b + 1, & \text{if } n = 0, \\ a, & \text{if } n = 1, b = 0, \\ 0, & \text{if } n = 2, b = 0, \\ 1, & \text{if } n \geq 3, b = 0, \\ H_{n-1}(a, H(a, b - 1)), & \text{otherwise.} \end{cases}$$

Example

H_1: addition; H_2: multiplication; H_3: exponentiation.

Hyperoperation

$$H_3(3,2): \quad 3 \wedge 2 = 3 \times (3 \wedge 1) = 3 \times (3 \times (3 \wedge 0)) = 3 \times (3 \times 1) = 3 \times 3 = 9$$

$$H_2(3,2): \quad 3 \times 2 = 3 + (3 \times 1) = 3 + (3 + (3 \times 0)) = 3 + (3 + 0) = 3 + 3 = 6$$

$$H_1(3,2): \quad 3 + 2 = 1 + (3 + 1) = 1 + (1 + (3 + 0)) = 1 + (1 + 3) = 1 + 4 = 5$$

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What about here, $H_{3/2}$?

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What about here, $H_{3/2}$?

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$$= 1 + (1 + 3) = 1 + 4 = 5$$

- What is $H_r(a,b), r \in \mathbb{R}_{\geq 0}$?

Neutral Element

What do we want?

1. Neutral element: ε_r
2.
3.

$$H_1 : \quad a + 0 = a$$
$$H_2 : \quad a \times 1 = a$$
$$H_3 : \quad a \wedge 1 = a$$

For H_r, we want a "right neutral element" ε_r so that $H_r(a, \varepsilon_r) = a$.

A "Fixed Point"

What do we want?

1. Neutral element: ε_r
2. $H_r(2,2) = 4$
3.

$$H_1 : \quad 2 + 2 = 4$$
$$H_2 : \quad 2 \times 2 = 4$$
$$H_3 : \quad 2 \wedge 2 = 4$$

In fact, $H_n(2,2) = 4$ for all $n \in \mathbb{N}$!
We want $H_r(2,2) = 4$ as well.

Mean Relation

What do we want?

1. Neutral element: ε_r
2. $H_r(2,2) = 4$
3. Mean function: m_r

$$
\begin{aligned}
H_1 : \quad & m_1(a,b) = (a+b)/2 \\
& \iff m_1(a,b) \times 2 = a + b \\
H_2 : \quad & m_2(a,b) = \sqrt{a \times b} \\
& \iff m_2(a,b) \wedge 2 = a \times b
\end{aligned}
$$

We want to have a "mean function" m_r, so that $H_{r+1}(m_r(a,b), 2) = H_r(a,b)$.

Playing around with m_r

Set $b := a$,

$$H_r(a, a) = H_{r+1}(m_r(a, a), 2)$$

$$\boxed{\implies H_r(a, a) = H_{r+1}(a, 2)}$$

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$a + a = a \times 2$,

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Set $b := a$,

$$H_r(a, a) = H_{r+1}(m_r(a, a), 2)$$

$$\boxed{\implies H_r(a, a) = H_{r+1}(a, 2)}$$

$a + a = a \times 2, a \times a = a \wedge 2, \ldots.$

Playing around with m_r

Set $a := H_{r+1}(a, 2), b := \varepsilon_r$,

$$\begin{aligned} H_{r+1}(m_r(H_{r+1}(a,2),\varepsilon_r),2) &= H_r(H_{r+1}(a,2),\varepsilon_r) \\ &= H_{r+1}(a,2) \end{aligned}$$

$$\boxed{\implies m_r(H_{r+1}(a,2),\varepsilon_r) = a}$$

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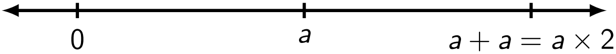
H_1:

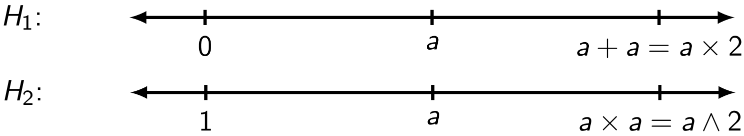


Playing around with m_r

Set $a := H_{r+1}(a, 2), b := \varepsilon_r$,

$$H_{r+1}(m_r(H_{r+1}(a,2), \varepsilon_r), 2) = H_r(H_{r+1}(a,2), \varepsilon_r)$$
$$= H_{r+1}(a, 2)$$

$$\boxed{\implies m_r(H_{r+1}(a,2), \varepsilon_r) = a}$$



Ground Rules

What do we want?

1. Neutral element: ε_r
2. $H_r(2,2) = 4$
3. Mean function: $m_r : H_{r+1}(m_r(a,b),2) = H_r(a,b)$
    - $H_r(a,a) = H_{r+1}(a,2)$
    - $m_r(H_{r+1}(a,2),\varepsilon_r) = a$

A Proposal for $m_{3/2}$

Definition (Minkowski Mean)

Let $p > 0$, $a, b \geq 0$, the Minkowski mean is

$$\mu_p(a, b) := \left(\frac{a^p + b^p}{2}\right)^{1/p}.$$

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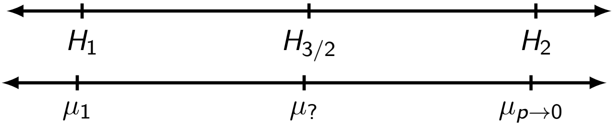
- If $p = 1$, then $\mu_1(a, b) = (a + b)/2$.
- If $p \to 0$, then $\mu_p(a, b) \to \sqrt{a \times b}$.

A Proposal for $m_{3/2}$



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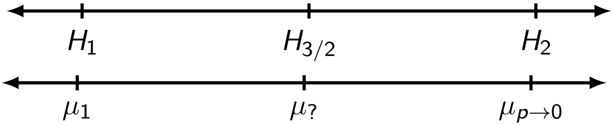

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How about $m_{3/2} := \mu_{1/2}$?

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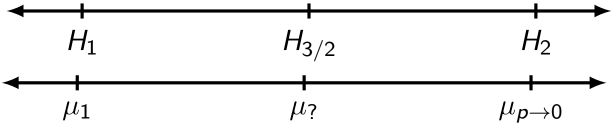

A Proposal for $m_{3/2}$

How about $m_{3/2} := \mu_{1/2}$?

$$M = \mu_{1/2}(a, b) = \left(\frac{a^{1/2} + b^{1/2}}{2}\right)^2 \iff \mu_{1/2}^{-1}(M, b) = (2M^{1/2} - b^{1/2})^2$$

Finding $\varepsilon_{3/2}$

$$\boxed{m_{3/2}(H_{5/2}(a,2),\varepsilon_{3/2}) = a}$$

Set $a := 2$,

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Set $a := 2$,

$$\begin{aligned} 2 &= \mu_{1/2}(H_{5/2}(2,2),\varepsilon_{3/2}) \\ &= \mu_{1/2}(4,\varepsilon_{3/2}) \end{aligned}$$

Finding $\varepsilon_{3/2}$

$$\boxed{m_{3/2}(H_{5/2}(a,2), \varepsilon_{3/2}) = a}$$

Set $a := 2$,

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$$\boxed{\implies \varepsilon_{3/2} = \mu_{1/2}^{-1}(2,4) \approx 0.6863}$$

Finding $H_{3/2}$

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$$\begin{aligned}\mu_{1/2}(a,b) &= \mu_{1/2}(H_{5/2}(\mu_{1/2}(a,b),2),\varepsilon_{3/2}) \\ &= \mu_{1/2}(H_{3/2}(a,b),\varepsilon_{3/2})\end{aligned}$$

Finding $H_{3/2}$

$$\boxed{m_{3/2}(H_{5/2}(a,2),\varepsilon_{3/2}) = a}$$

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$$\boxed{\implies H_{3/2}(a,b) = \mu_{1/2}^{-1}(\mu_{1/2}(a,b),\varepsilon_{3/2})}$$

Calculate $H_{3/2}$

$a \setminus b$	0	1	2	3	4	5
0	0.6863	0.0294	0.3431	0.8165	1.3726	1.9815
1	0.0294	1.3726	2.5147	3.6238	4.7157	5.7967
2	0.3431	2.5147	4	5.3724	6.6863	7.9629
3	0.8165	3.6238	5.3724	6.9468	8.431	9.8577
4	1.3726	4.7157	6.6863	8.431	10.0589	11.612
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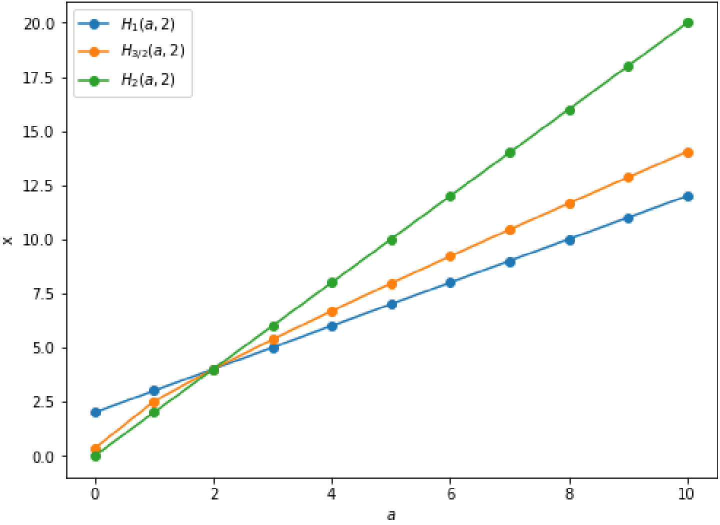

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- $H_{5/2}$, between $\times$ and $\wedge$?

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  $H_{5/2}(a, b+1) = H_{3/2}(a, H_{5/2}(a, b))$.

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$$
\begin{aligned}
H_{3/2}(a, a) &= H_{5/2}(a, 2) \\
&= H_{3/2}(a, H_{5/2}(a, 1)) \\
&\boxed{\implies H_{5/2}(a, 1) = a}
\end{aligned}
$$

...And Beyond!

- $H_{5/2}$, between $\times$ and $\wedge$? Induction!
  $H_{5/2}(a, b+1) = H_{3/2}(a, H_{5/2}(a, b))$.

$$\begin{aligned} H_{3/2}(a, a) &= H_{5/2}(a, 2) \\ &= H_{3/2}(a, H_{5/2}(a, 1)) \\ &\boxed{\implies H_{5/2}(a, 1) = a} \\ H_{3/2}(a, \varepsilon_{3/2}) &= a \\ &= H_{3/2}(a, H_{5/2}(a, 1)) \\ &\boxed{\implies H_{5/2}(a, 0) = \varepsilon_{3/2}} \end{aligned}$$

...And Beyond!

$a \setminus b$	0	1	2	3	4	5
0	0.6863	0	0.6863	0	0.6863	0
1	0.6863	1	1.3726	1.804	2.2944	2.8436
2	0.6863	2	4	6.6863	10.0589	14.1177
3	0.6863	3	6.9468	12.5266	19.7396	28.5855
4	0.6863	4	10.0589	18.8629	30.4121	44.7065
5	0.6863	5	13.2766	25.5161	41.7186	61.8839

...And Beyond!