Vagenende, Brando; **Verbeken, Brecht**; **Guerry, Marie-Anne**
Eigenvalue regions and realising monotone stochastic matrices. (English) Zbl 08205034
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A stochastic matrix is a nonnegative matrix whose rows sum to one. The spectral theory of such matrices has been extensively developed, largely because of their importance in Markov chain theory. One may study individual eigenvalues, leading to the classical Karpelevich description of the possible eigenvalue regions, or study the spectrum as a whole, an approach connected with inverse eigenvalue problems and results such as the Johnson-Loewy-London inequalities.

Similar questions have been considered for various subclasses of stochastic matrices, including doubly stochastic matrices. The present paper deals with monotone stochastic matrices in the sense of Daley, namely stochastic matrices $M = (m_{ij})$ satisfying

$$\sum_{j=r}^{n} m_{lj} \geq \sum_{j=r}^{n} m_{kj}, \qquad l > k, \quad r = 1, \dots, n.$$

In other words, the cumulative tail sums of the rows increase from top to bottom.

The main tool is the associated dominance matrix, a nonnegative matrix of order one less than the original matrix whose spectrum consists precisely of the nontrivial eigenvalues of the monotone matrix. The authors establish several basic properties of dominance matrices and characterize those 2×2 nonnegative matrices that arise from 3×3 monotone stochastic matrices.

They then determine completely the eigenvalue regions Ξ_n for $n = 1, 2, 3$, proving that

$$\Xi_1 = \{1\}, \qquad \Xi_2 = [0, 1], \qquad \text{and} \qquad \Xi_3 = [-1/2, 1].$$

Explicit realizing matrices are given in each case. These regions are strictly smaller than the corresponding Karpelevich regions for general stochastic matrices.

The paper also describes all possible pairs of nontrivial eigenvalues of a 3×3 monotone stochastic matrix by determining the boundary of the corresponding realization region and providing realizing matrices. Finally, it is shown that for every $n \geq 3$, the eigenvalue region of $n \times n$ monotone stochastic matrices is contained in the Karpelevich region of $(n-1) \times (n-1)$ stochastic matrices.

Reviewer: Frédéric Morneau-Guérin (Québec)

MSC:

15B51 Stochastic matrices
15A18 Eigenvalues, singular values, and eigenvectors
15A29 Inverse problems in linear algebra
15A42 Inequalities involving eigenvalues and eigenvectors

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