A Complete Solution to a Conjecture on Doubly Stochastic Eigenvalues

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Stochastic Matrices

Definition

A square matrix S is called stochastic if

1. *its entries are nonnegative,*
2. *the sum of each of its rows is equal to 1.*

The set of stochastic matrices of order n is denoted by $\mathbb{S}_n$.

Example (Stochastic matrices of order 3)

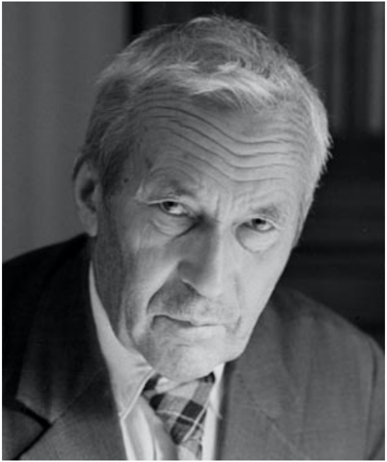
$$S = \begin{pmatrix} 0.7 & 0.1 & 0.2 \\ 1 & 0 & 0 \\ 0.5 & 0.4 & 0.1 \end{pmatrix}, \qquad P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

A Spectral Problem

• **1938 :** Kolmogorov propose the problem of characterizing the set

$$\Omega_n := \bigcup_{S\in\mathbb{S}_n}\sigma(S)$$

of all eigenvalues of all stochastic matrices for every $n \geq 1$.



Andrey Kolmogorov

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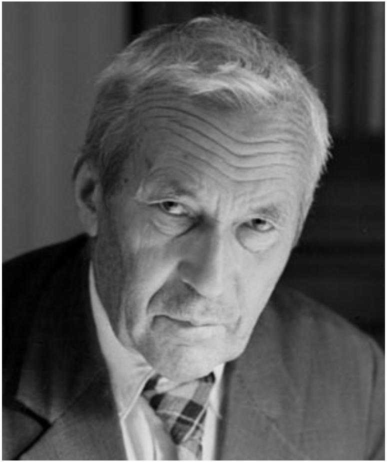
- **1951 :** Karpelevič obtains a complete description of Ω_n for all $n \geq 1$.



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Karpelevič's Result

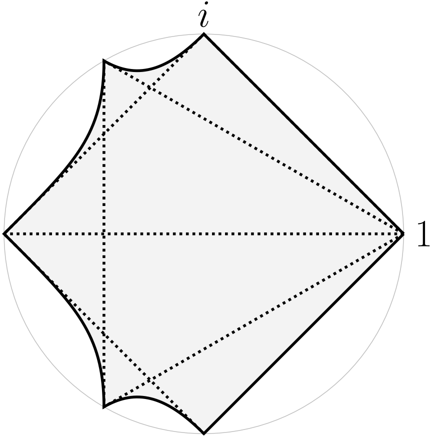
Theorem (Karpelevič, 1951 ; Ito, 1997)

The region Ω_n is symmetric with respect to the real axis, is contained in the unit disk $|z| \leq 1$, and intersects the unit circle $|z| = 1$ at the points $e^{2\pi i a/b}$, where a and b are relatively prime integers satisfying $0 \leq a \leq b \leq n$. The curve delimiting the boundary of Ω_n consists of these points and curvilinear arcs joining them in counterclockwise order. If the endpoints of an arc are $e^{2\pi i a_1/b_1}$ and $e^{2\pi i a_2/b_2}$ $(b_1 \leq b_2)$, then the arc is characterized by the parametric equation

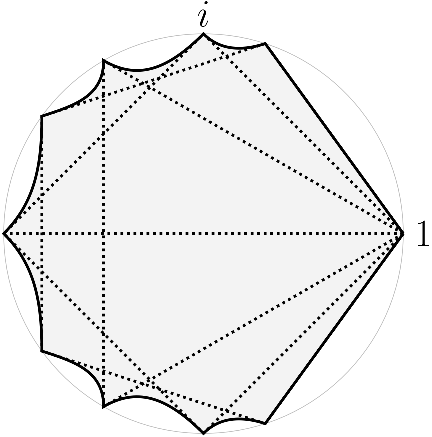
$$\lambda^{b_2}(\lambda^{b_1} - s)^{\lfloor n/b_1 \rfloor} = (1 - s)^{\lfloor n/b_1 \rfloor} \lambda^{b_1 \lfloor n/b_1 \rfloor},$$

where $s \in [0, 1]$ and $\lfloor \cdot \rfloor$ is the floor function.

The Karpelevič Regions



The region Ω_4

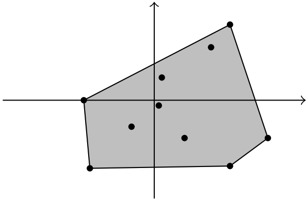


The region Ω_5

Proof Idea

Theorem (Dmitriev, Dynkin, 1946)

Let $\vec{z} \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$. Set $K_1 := \text{Conv}\{z_1, z_2, \ldots, z_n\}$. Then there exists a stochastic matrix $S \in \mathbb{S}_n$ such that $S\vec{z} = \lambda\vec{z}$ (i.e., $\lambda \in \Omega_n$) if and only if $\lambda K_1 \subseteq K_1$.



$K_1 = \text{Conv}\{z_1, z_2, \ldots, z_n\}$

Doubly Stochastic Matrices

Definition

A square matrix D is called <u>*doubly stochastic*</u> *if*

1. *its entries are nonnegative,*
2. *the sum of each of its rows is equal to* 1*,*
3. *the sum of each of its columns is equal to* 1*.*

The set of doubly stochastic matrices of order n is denoted by $\mathbb{D}_n$*.*

Example (Doubly stochastic matrices of order 3)

$$D = \begin{pmatrix} 0.7 & 0.1 & 0.2 \\ 0 & 0.5 & 0.5 \\ 0.3 & 0.4 & 0.3 \end{pmatrix}, \qquad P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Main Problem

Question (Mirsky, 1963)

For which numbers $\lambda \in \mathbb{C}$ is there an $n \times n$ doubly stochastic matrix D such that λ is an eigenvalue of D ? That is, characterize the set

$$\omega_n := \{\lambda \in \mathbb{C} : \exists D \in \mathbb{D}_n \text{ s.t. } \lambda \in \sigma(D)\} = \bigcup_{D \in \mathbb{D}_n} \sigma(D).$$

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  $\Longrightarrow$ We may characterize ω_n by the polar equation of its boundary.

- We have the inclusions $\omega_n \subseteq \Omega_n \subseteq \overline{\mathbb{D}}$. In particular, ω_n is contained in the closed unit disk $\overline{\mathbb{D}}$.

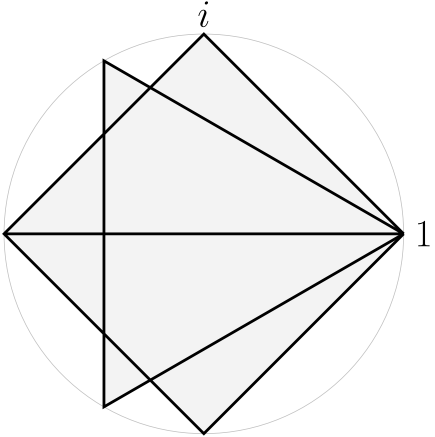
A Set Inclusion

Theorem (Perfect, Mirsky, 1965)

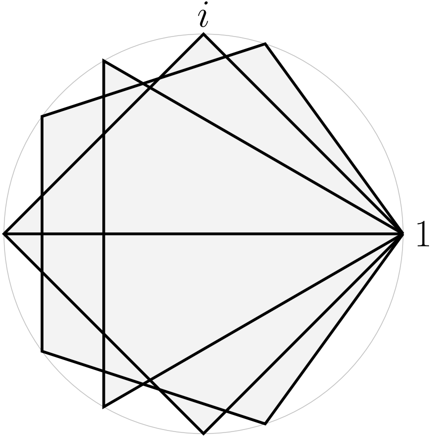
Let $\zeta_k := e^{2\pi i/k}$ *and* $\Pi_k := \mathrm{Conv}\{1, \zeta_k, \zeta_k^2, \ldots, \zeta_k^{k-1}\}$*. Then, for* $n \geq 1$*,*

$$\Pi_1 \cup \Pi_2 \cup \cdots \cup \Pi_n \subseteq \omega_n.$$

Some Examples



The region $\cup_{k=1}^{4}\Pi_k$



The region $\cup_{k=1}^{5}\Pi_k$

The Perfect–Mirsky Conjecture

Conjecture (Perfect, Mirsky, 1965)

$$\omega_n = \bigcup_{k=1}^{n} \Pi_k =: PM_n.$$

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- $n = 2$: True
- $n = 3$: True (Perfect, Mirsky, 1965).

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- $n = 1$: True
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- $n = 3$: True (Perfect, Mirsky, 1965).
- $n = 4$: True (Levick, Pereira, Kribs, 2015).

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- $n = 1$: True
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- $n = 3$: True (Perfect, Mirsky, 1965).
- $n = 4$: True (Levick, Pereira, Kribs, 2015).
- $n = 5$: *False* (Mashreghi, Rivard, 2007).

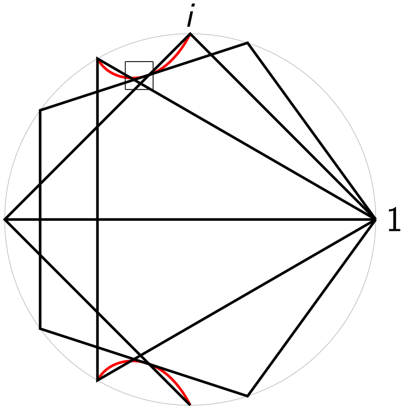
The Counterexample

The doubly stochastic matrices

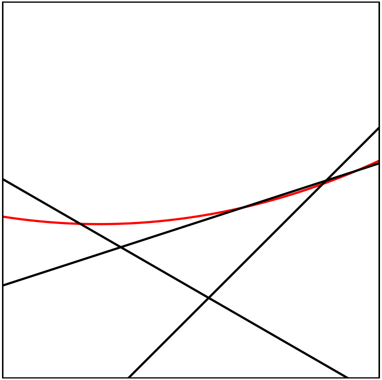
$$t\begin{pmatrix} 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} \\ \mathbf{1} & 0 & 0 & 0 & 0 \end{pmatrix} + (1-t)\begin{pmatrix} 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 & 0 \end{pmatrix}$$

have eigenvalues outside $\bigcup_{k=1}^{5} \Pi_k$ for $t \in [0.4705275, 0.5490013]$.

The Exceptional Curve



The Perfect–Mirsky region $\bigcup_{k=1}^{5} \Pi_k$ and the exceptional curve of Mashreghi–Rivard.



Zoom on the exceptional curve around $[-0.35, -0.2] \times [0.7, 0.85]$.

Birkhoff's Theorem

Theorem (Birkhoff's theorem, 1946)

The set $\mathbb{D}_n$ of $n \times n$ doubly stochastic matrices is the convex hull of the permutation matrices of order n. Moreover, the permutation matrices are precisely the extreme points of $\mathbb{D}_n$.

What about the cases $n \geq 6$?

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- However, recent work (2022) by Harlev, Johnson, and Lim seems to support the validity of the conjecture.

Majorization

Definition

Let $x_1^{\downarrow} \geq \cdots \geq x_n^{\downarrow}$ be the decreasing rearrangement of the entries of $x = (x_1, \ldots, x_n)$. Then y is majorized by x, and we write $y \prec x$, if

$$\sum_{j=1}^{k} y_j^{\downarrow} \leq \sum_{j=1}^{k} x_j^{\downarrow}, \qquad k = 1, 2, \ldots, n,$$

with equality when $k = n$.

Example

$$\left(\frac{1}{n}, \ldots, \frac{1}{n}\right) \prec (a_1, \ldots, a_n) \prec (1, 0, \ldots, 0)$$

for all $a_j \geq 0$ $(1 \leq j \leq n)$ such that $\sum_{j=1}^{n} a_j = 1$.

The Hardy, Littlewood and Pólya

Theorem (HLP, 1929)

Let $x, y \in \mathbb{R}^n$. The following statements are equivalent :

1. $y = Dx$ *for some doubly stochastic matrix* D *;*
2. $y \prec x$.

A Geometric Property

Definition

Let $\vec{z} \in \mathbb{C}^n$. We then define the polygons

$$K_1 := \mathrm{Conv}\{z_1, z_2, \ldots, z_n\}$$

$$K_2 := \mathrm{Conv}\left\{\frac{z_i + z_j}{2} : i \neq j\right\}$$

$$\vdots$$

$$K_m := \mathrm{Conv}\left\{\frac{z_{i_1} + z_{i_2} + \cdots + z_{i_m}}{m} : i_j \neq i_k \text{ for all } j \neq k\right\}$$

$$\vdots$$

$$K_n := \mathrm{Conv}\left\{\frac{z_1 + z_2 + \cdots + z_n}{n}\right\}.$$

A Geometric Property

Theorem (B., Mashreghi, Morneau-Guérin, 2026+)

If $\lambda \in \omega_n$, then there exists $\vec{z} \in \mathbb{C}^n$ such that

$$\lambda K_m \subseteq K_m \quad \textit{for all } m = 1, 2, \ldots, n.$$

A Difficult Problem

Theorem (Dmitriev, Dynkin, 1946)

We have $\lambda \in \Omega_n$ if and only if there exists $\vec{z} \in \mathbb{C}^n$ such that

$$\lambda K_1 \subseteq K_1.$$

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An Interesting Special Case

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Let $\vec{z} \in \mathbb{C}^n$. We say that $\vec{z}$ is convexly independant if each of its entries is not a convex combination of the other $n-1$ entries.

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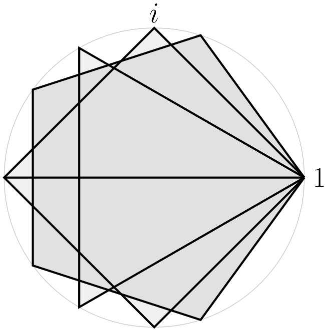
Conjecture (Levick, Pereira, Kribs, 2015)

If $\lambda \in \mathbb{C}$ is an eigenvalue of a doubly stochastic matrix associated with a convexly independant eigenvector, then $\lambda \in PM_n$.

A Positive Result

Theorem (B., Mashreghi, Morneau-Guérin; 2026+)

If $\lambda \in \mathbb{C}$ is an eigenvalue of an $n \times n$ doubly stochastic matrix associated to a convexly independent eigenvector, then $\lambda \in \Pi_n \subsetneq PM_n$.



Reformulation with Circulant Matrices

Observation: If $\vec{z}$ is convexly independent, then $K_2, K_3, \ldots,$ have a simple form.

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Proposition (B., Mashreghi, Morneau-Guérin; 2026+)

If $\vec{z} \in \mathbb{C}^n$ is convexly independent with the z_j in clockwise order, then the extreme points of K_m are given in clockwise order by $T_m\vec{z}$, where T_m is the circulant matrix

$$T_m := \operatorname{circ}\big(\underbrace{1, \ldots, 1}_{m}, \underbrace{0, \ldots, 0}_{n-m}\big).$$

Maximal Eigenvectors

Definition

An eigenpair $(re^{i\theta}, \vec{z})$ of an $n \times n$ doubly stochastic matrix, with $\vec{z}$ convexly independent, is called *maximal* if $(r + \varepsilon)e^{i\theta}$ is not an eigenvalue of any $n \times n$ doubly stochastic matrix with a convexly independent eigenvector.

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Proposition (B., Mashreghi, Morneau-Guérin; 2026+)

If $(\lambda, \vec{z})$ is a maximal eigenpair, then

$$T_2\vec{z} = \left(\frac{z_1 + z_2}{2}, \frac{z_2 + z_3}{2}, \ldots, \frac{z_{n-1} + z_n}{2}, \frac{z_n + z_1}{2}\right)$$

is also a maximal eigenvector (associated to λ).

A Dynamical Approach

Corollary (B., Mashreghi, Morneau-Guérin; 2026+)

If $(\lambda, \vec{z})$ is a maximal eigenpair, then $\rho^k T_2^k \vec{z}$ is also a maximal eigenpair (associated to λ) for all $\rho > 0$ and all $k \in \mathbb{N}$.

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Consequence : Letting $k \to \infty$ in the corollary and using a continuity argument, we may suppose without loss of generality that $\vec{z}$ satisfy

$$z_j = \alpha e^{\frac{2\pi ij}{n}} + \beta e^{\frac{-2\pi ij}{n}}, \qquad j = 1, 2, \ldots, n,$$

for some $\alpha, \beta \in \mathbb{C}$.

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for some $\alpha, \beta \in \mathbb{C}$.

Maximizing $|\lambda|$ for the eigenvectors of this form yield the desired result. $\square$

Consequence for the Case $n = 5$

Theorem (B., Mashreghi, Morneau-Guérin; 2026+)

If $(\lambda, \vec{z})$ is an eigenpair of a 5×5 doubly stochastic matrix which is not in PM_5, then $K_1 = \mathrm{Conv}\{z_1, z_2, z_3, z_4, z_5\}$ has exactly 4 extreme points.

Heuristic for the cases $n \geq 6$

- The question can be reformulated as a constrained optimization problem.

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- The question can be reformulated as a constrained optimization problem.

$$\begin{cases} \text{Number of variables} \approx 2n \\ \text{Number of constraints} \approx n\lfloor \frac{n}{2} \rfloor \end{cases} \implies \begin{cases} \text{variable} \geq \text{constraint} & \text{if } n \leq 5 \\ \text{variable} < \text{constraint} & \text{if } n > 5 \end{cases}$$

Consequence : Too many constraints force the solution into the local minimum that is the Perfect–Mirsky regions for $n > 5$.

Explanation for the cases $n \leq 4$

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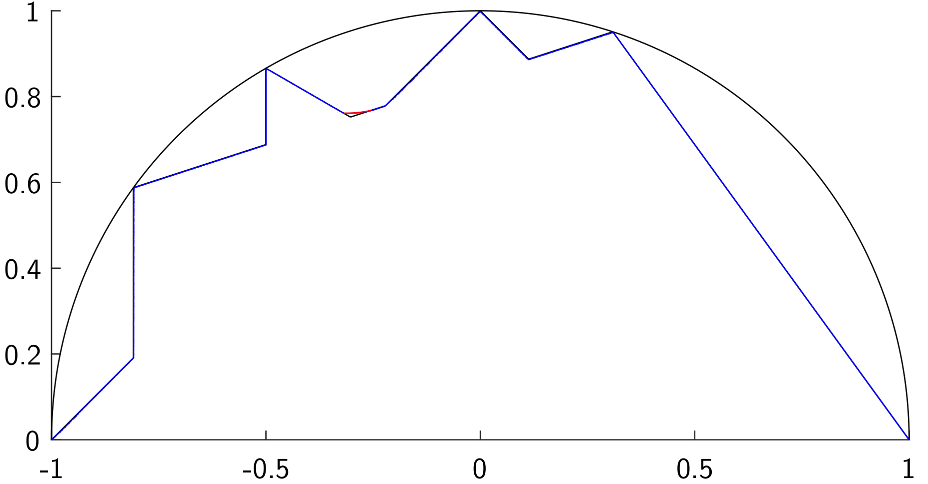
$$\lambda \in \omega_n \quad \implies \quad \lambda K_1 \subseteq K_1 \quad \implies \quad \lambda \in \Omega_m = \omega_m.$$

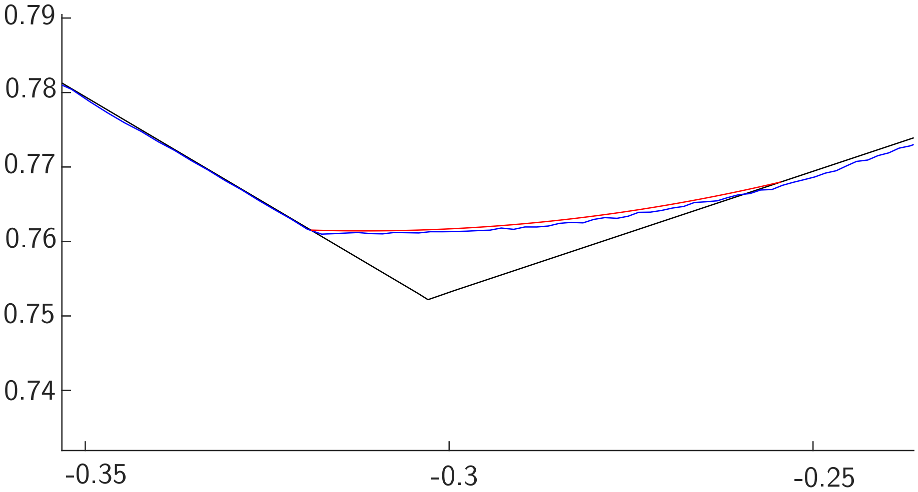
- If $m = n = 4$, then our result shows that $\lambda \in \Pi_4$.

Numerical Computations



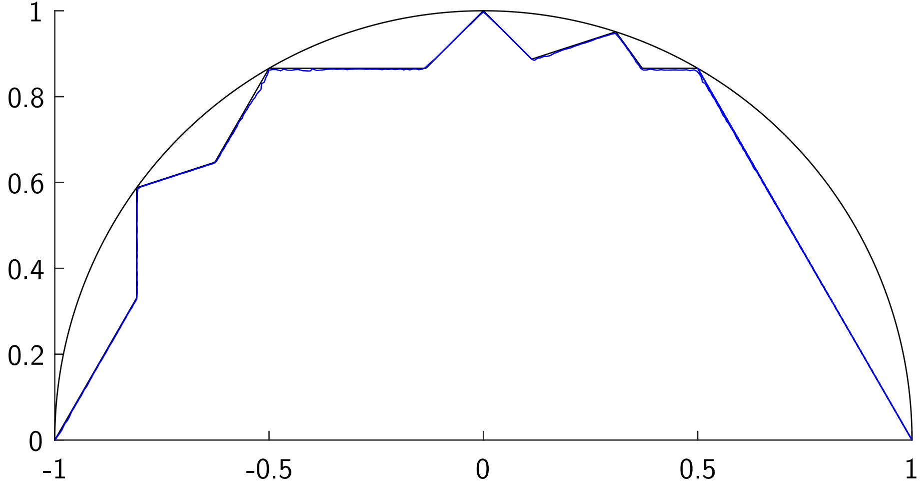
Approximation of the optimal radius for $n = 5$

Numerical Computations



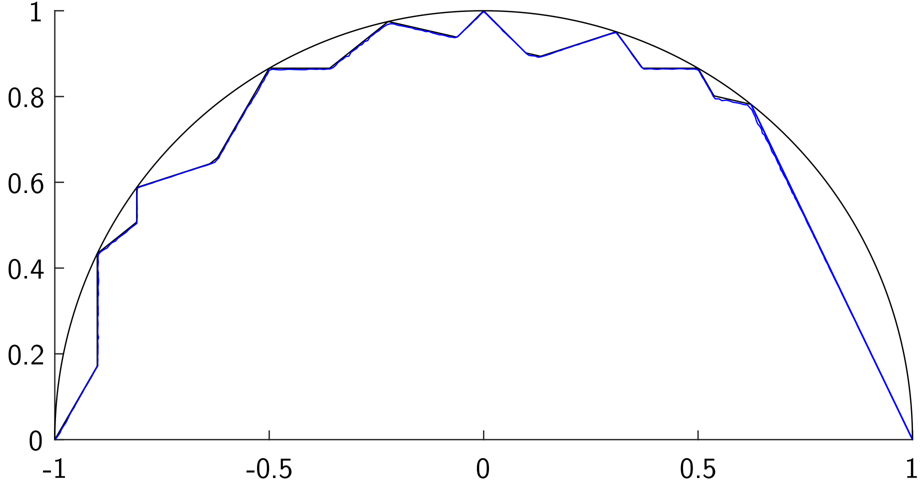
Zoom on the exceptional region of the approximation of the optimal radius

Numerical Computations



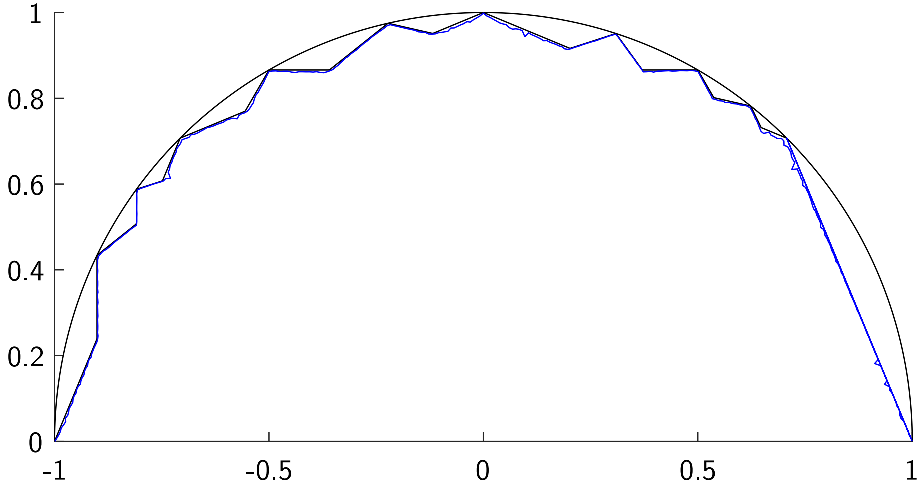
Approximation of the optimal radius for $n = 6$

Numerical Computations



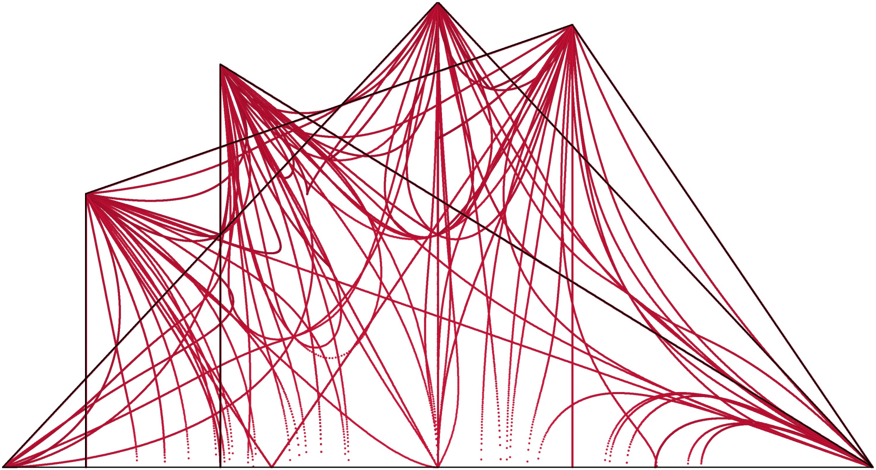
Approximation of the optimal radius for $n = 7$

Numerical Computations



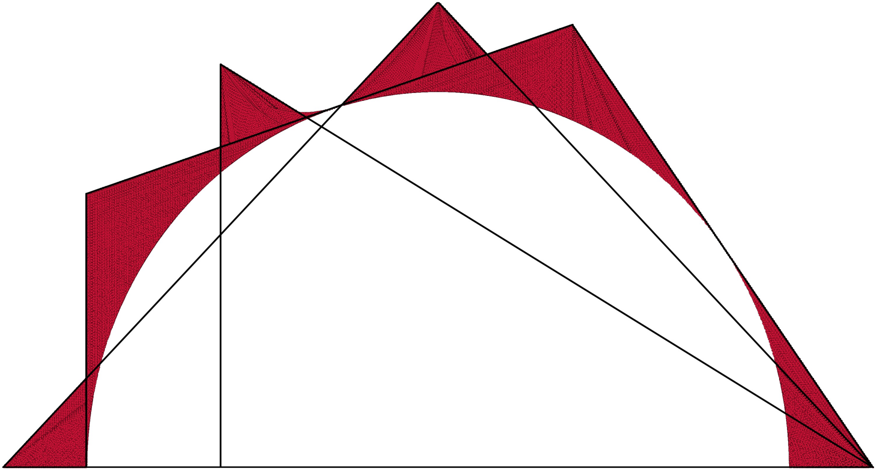
Approximation of the optimal radius for $n = 8$

Some nice figures



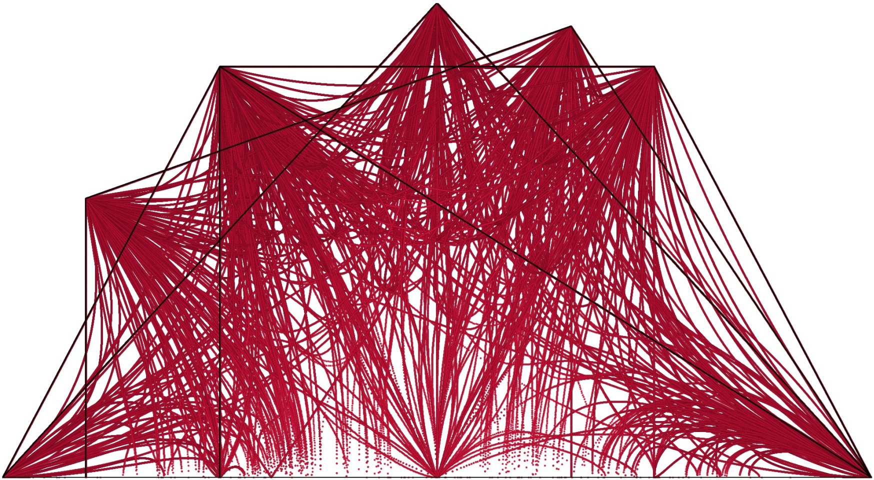
Eigenvalues of convex combinations of pairs of permutation matrices for $n = 5$.

Some nice figures



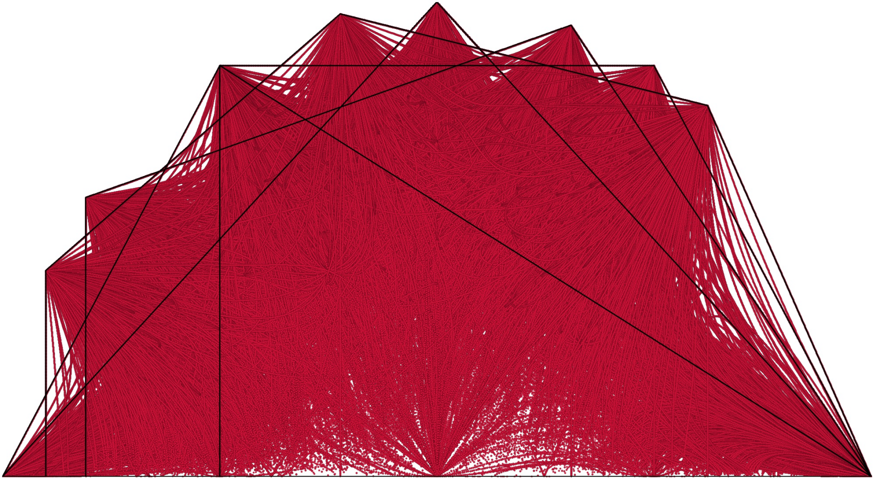
Eigenvalues of convex combinations of triples of permutation matrices for $n = 5$; the inscribed circle has been omitted.

Some nice figures



Eigenvalues of convex combinations of pairs of permutation matrices for $n = 6$.

Some nice figures



Eigenvalues of convex combinations of pairs of permutation matrices for $n = 7$.

Harlev, Johnson, and Lim conjectures

Conjecture (Harlev–Johnson–Lim, 2022)

$\omega_n = PM_n$ *for* $n = 6, 7, 8, 9, 10, 11$.

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Conjecture (Boundary conjecture, 2022)

For $n \geq 1$*, pairs of permutation matrices determine the boundary of* ω_n*. That is, each point of* $\partial\omega_n$ *is an eigenvalue of a convex combination of at most two permutation matrices.*