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Characterization of Erdős matrices by their zero entries. (English) Zbl 08184654
Linear Algebra Appl. 739, 154-169 (2026).

A doubly stochastic matrix (also called bistochastic matrix) is a square matrix of nonnegative real numbers, with all row and column sums equal to 1.

In [Q. J. Math., Oxf. II. Ser. 10, 296–302 (1959; Zbl 0138.01501)], *M. Marcus* and *R. Ree* proved that any $n \times n$ doubly stochastic matrix A satisfies

$$\sum_{i=1}^n \sum_{j=1}^n a_{i,j}^2 \leq \max_{\sigma \in S_n} \sum_{i=1}^n A_{i,\sigma(i)},$$

where S_n is the set of all permutations of the integers $1, 2, \dots, n$. To put it another way, the Marcus-Ree inequality says that any doubly stochastic matrix has a maximal trace (or maximal diagonal sum) greater than or equal to its Frobenius norm squared.

Seeing this inequality, Erdős asked for which doubly stochastic matrices it is saturated. Such matrices will henceforth be referred to as Erdős matrices. Since both the functions

$$A \mapsto \|A\|_F^2 := \sum_{i=1}^n \sum_{j=1}^n a_{i,j}^2 \text{ and } A \mapsto \max \text{Trace}(A) := \max_{\sigma \in S_n} \sum_{i=1}^n A_{i,\sigma(i)}$$

are invariant under left and right multiplication by permutation matrices, Erdős matrices are considered up to this equivalence.

The known results may be summarized as follows: for $n = 2$ there are 2 equivalence classes; for $n = 3$ there are 6 classes [*L. Bouthat* et al., Linear Multilinear Algebra 72, No. 17, 2823–2844 (2024; Zbl 1555.15036)]; for $n = 4$ there are 41 classes [*A. Kushwaha* and *R. Tripathi*, Linear Algebra Appl. 725, 223–247 (2025; Zbl 08091342)]; for general n there are finitely many Erdős matrices [*R. Tripathi*, Linear Algebra Appl. 708, 236–251 (2025; Zbl 1562.15032)], all with rational entries.

The present paper studies Erdős matrices via their zero patterns (called *skeletons*). It is shown that each skeleton supports at most one Erdős matrix, so that Erdős matrices are uniquely determined by their zero entries. This reduces their classification to the study of binary matrices with total support.

A key structural result is that every Erdős matrix is a restricted common diagonal sum (RCDS) matrix, i.e., all diagonal sums corresponding to permutations compatible with its zero pattern are equal. Conversely, an RCDS bistochastic matrix is Erdős if and only if this common value coincides with its maximal diagonal sum.

Using this framework, the authors derive a characterization of Erdős matrices in terms of their skeletons and establish monotonicity properties relating the inclusion of zero patterns to the maximal trace.

An algorithm based on skeleton enumeration and linear algebra is proposed to generate all Erdős matrices. This method is used to compute all such matrices up to size 6×6 , yielding 469 equivalence classes for $n = 5$ and 23 851 for $n = 6$.

Additional results include constructions of new families of Erdős matrices, connections with transportation polytopes, and extremal examples with respect to the number of distinct entries and denominator size.

Reviewer: [Frédéric Morneau-Guérin](#) (Québec)

MSC:

- 15B51 Stochastic matrices
- 15A15 Determinants, permanents, traces, other special matrix functions
- 15A45 Miscellaneous inequalities involving matrices
- 15B36 Matrices of integers

Keywords:

bistochastic matrix; Frobenius norm; skeleton; inner and outer (permutations and) traces; RCDS matrix; Erdős matrix

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