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Cao, Lei; Eshkaftaki, Ali Bayati; Koyuncu, Selcuk

Linear preservers of sub-defect of doubly substochastics matrices. (English) Zbl 08179490

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A square matrix with nonnegative entries is called *doubly stochastic* if all row and column sums equal 1, and *doubly substochastic* if all row and column sums are ≤ 1. Denote by Ω_n and ω_n the corresponding polytopes of $n \times n$ matrices.

The polytope Ω_n (the Birkhoff polytope) is the convex hull of permutation matrices, while ω_n is the convex hull of partial permutation matrices, i.e., $(0,1)$-matrices with at most one nonzero entry in each row and column.

L. Cao et al. [Linear Multilinear Algebra 64, No. 11, 2313–2334 (2016; Zbl 1358.15025)] introduced the concept of the *sub-defect* of an $n \times n$ doubly substochastic matrix B, denoted by $sd(B)$, which is defined as the minimum number of rows and columns needed to be added to B to obtain a doubly stochastic matrix A containing B as a submatrix, Moreover, they showed that the sub-defect of B can be calculated using the sum of all elements of B, i.e.,

$$sd(B) = \left\lceil n - \sum_{i=1}^{n} \sum_{j=1}^{n} b_{i,j} \right\rceil.$$

The paper characterizes linear operators $T : M_m \to M_n$ preserving the sub-defect, i.e.,

$$T(\omega_m) \subseteq \omega_n, \qquad sd(T(A)) = sd(A) \text{ for all } A \in \omega_m.$$

It is shown that such operators exist only for $m = n$, and that T preserves the total sum of entries. Moreover, T admits a representation

$$T(A) = \sum_{i,j} a_{ij} s_{ij},$$

where each s_{ij} is a nonnegative matrix of total sum 1, and T maps partial permutation matrices into ω_n. This yields a complete characterization of linear preservers of the sub-defect.

Reviewer: Frédéric Morneau-Guérin (Québec)

MSC:

15B51 Stochastic matrices
15A86 Linear preserver problems
47A57 Linear operator methods in interpolation, moment and extension problems

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