MR4940354 15A60 46B20

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Best constants for the l^p inequalities associated to some particular matrices. (English. English summary)

Results Math. **80** (2025), *no.* 6, *Paper No.* 176, 19 *pp.*

Suppose a vector norm $\|\cdot\|_\alpha$ on $\mathbb{R}^n$ and a vector norm $\|\cdot\|_\beta$ on $\mathbb{R}^m$ are given. Any $m\times n$ matrix A induces a linear operator from $\mathbb{R}^n$ to $\mathbb{R}^m$, and one defines the corresponding *induced norm* (or *operator norm*) on the space of all $m\times n$ matrices as

$$\|A\|_{\alpha,\beta} := \sup_{x\neq 0} \frac{\|Ax\|_\beta}{\|x\|_\alpha}.$$

In what follows, we restrict our attention to the case where the vector norms on $\mathbb{R}^n$ and $\mathbb{R}^m$ are ℓ^p-norms (recall that, for $1\le p<\infty$, $\|x\|_p = (\sum_i |x_i|^p)^{1/p}$, while $\|x\|_\infty = \max_i |x_i|$).

While the induced norms $\|A\|_{1,1}$, $\|A\|_{\infty,\infty}$ and $\|A\|_{2,2}$ admit explicit expressions—namely, the maximum absolute column sum, the maximum absolute row sum, and the largest singular value of A, respectively—computing $\|A\|_{p,q}$ for $(p,q)\neq(1,1),(\infty,\infty),(2,2)$ is in general a difficult problem.

In the article under consideration, the authors present explicit formulas for the ℓ^p operator norms of several structured families of matrices.

The first family considered consists of circulant matrices with nonnegative entries. Building on known special cases and using their spectral decomposition together with Riesz-Thorin interpolation, the authors show that for such matrices $\|\cdot\|_{p,p}$ is independent of p for $2\le p\le\infty$ and coincides with the spectral radius.

The authors then extend these results to matrices with constant line sums. Indeed, they show that for symmetric matrices, and more generally for matrices with both constant row and column sums, $\|\cdot\|_{p,p}$ is again independent of p for $2\le p\le\infty$. As a consequence, all doubly stochastic matrices are shown to verify $\|\cdot\|_{p,p}=1$ for $2\le p\le\infty$. The same approach also applies to diagonal scalings of such matrices, yielding exact norm computations even in the presence of zero or negative entries.

The authors further extend their results to block diagonal matrices with constant line-sum blocks, showing that $\|\cdot\|_{p,p}$ is governed by the dominant block. They also present explicit families of rectangular and block matrices for which the $\|\cdot\|_{p,p}$ genuinely depends on p and derive closed-form expressions in these cases.

Finally, the authors offer concluding remarks and perspectives, including connections with subnorms of invertible matrices, Sinkhorn's theorem, and possible extensions to skew-symmetric and tridiagonal matrices, motivated in part by links with discrete Hilbert transforms and related operator-theoretic problems.

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.