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On a class of infinite M-matrices with an extended quasi-Toeplitz structure. (English)

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Over the past decades, the theoretical analysis of finite M-matrices, together with the computation of square roots of nonsingular M-matrices, has been extensively studied. Their structural properties, as well as the algorithmic issues they raise and their numerous applications, are well documented in the literature.

The theory of M-matrices has subsequently been extended to more general settings, notably to certain nonassociative structures such as Euclidean Jordan algebras and, more generally, to M-operators in partially ordered real Banach spaces.

The study of quasi-Toeplitz M-matrices originates from that of quasi-Toeplitz matrices, which arise in a variety of applications in numerical analysis and stochastic modeling.

A quasi-Toeplitz matrix can be written as the sum of a semi-infinite Toeplitz matrix and a correction matrix. Two important classes are commonly considered, namely the quasi-Toeplitz class $\mathcal{QT}_\infty$ and the extended quasi-Toeplitz class $\mathcal{EQT}_\infty$, both of which are Banach algebras.

Quasi-Toeplitz M-matrices in the class $\mathcal{QT}_\infty$ were studied in previous work by the same author, where necessary and sufficient conditions for the M-matrix property were obtained. It was also shown that invertible matrices in this class admit a unique square root that is itself an invertible quasi-Toeplitz M-matrix.

The present paper extends this framework to the larger class $\mathcal{EQT}_\infty$, which can be viewed as a rank-one perturbation of quasi-Toeplitz matrices. Such matrices arise, for instance, in Markov processes involving restart mechanisms. Their structural properties have so far received comparatively little attention.

The main contribution is to develop a theoretical and computational treatment of M-matrices in $\mathcal{EQT}_\infty$. In particular, the author establishes conditions ensuring existence and uniqueness of square roots within this class.

On the computational side, the paper proposes algorithms that exploit the matrix structure for the efficient computation of the square root, including adaptations of classical iterations (such as binomial iteration and cyclic reduction) as well as structure-preserving methods.

A further contribution concerns the analysis of a nonlinear equation of the form

$$Vx - \mu_x x \;=\; b,$$

which arises in the characterization of the extended component of the square root. Existence, uniqueness, and convergence results are obtained, together with iterative schemes for its numerical solution.

The paper is organized as follows:

Section 2 recalls preliminary notions on extended quasi-Toeplitz matrices and M-operators in Banach spaces.

Section 3 studies structural properties of M-matrices in $\mathcal{EQT}_\infty$ and establishes existence and uniqueness results for their square roots.

Section 4 develops algorithms for computing these square roots.

Section 5 analyzes the associated nonlinear equation.

Section 6 presents numerical experiments.

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MSC:

15B05 Toeplitz, Cauchy, and related matrices
15A24 Matrix equations and identities
15A16 Matrix exponential and similar functions of matrices
65F45 Numerical methods for matrix equations
65F60 Numerical computation of matrix exponential and similar matrix functions

Keywords:

extended quasi-Toeplitz matrix; infinite M-matrix; structured-preserving doubling algorithm; system of nonlinear equations

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