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New norm and numerical radius bounds via advanced Cauchy-Schwarz inequalities. (English)

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The Cauchy-Schwarz inequality, which states that in an inner product space $\big(X, \langle\cdot,\cdot\rangle\big)$,

$$|\langle x, y\rangle| \le \|x\|\,\|y\|, \qquad (x, y \in X),$$

is one of the most fundamental and widely used inequalities in mathematics.

In the setting of Hilbert spaces, this inequality extends naturally to operator expressions. More precisely, if $\mathbb{H}$ is a Hilbert space and $\mathbb{B}(\mathbb{H})$ denotes the C^*-algebra of bounded linear operators on $\mathbb{H}$ with identity I, then

$$|\langle Tx, y\rangle| \le \|Tx\|\,\|y\|, \qquad (x, y \in \mathbb{H},\ T \in \mathbb{B}(\mathbb{H})).$$

In the present paper, the authors establish new inner product inequalities of Cauchy-Schwarz type, in which expressions involving multiple operators (typically sums such as $A + B$) are controlled within a single inequality (so-called mixed Cauchy-Schwarz inequalities). These refinements are then used to derive bounds for operator expressions involving unitarily invariant norms and the numerical radius.

After a brief introduction, *Section 1* places the Cauchy-Schwarz-type inequalities in context and gathers several auxiliary lemmas from the literature that are used throughout the paper.

Section 2 contains the main contributions: new inner product inequalities together with their proofs and a number of remarks illustrating their consequences and their relation to existing bounds.

Section 3 is devoted to applications. The authors derive bounds for operator norms and numerical radii in two settings: first for expressions of the form $X(A + B)Y$, and then, as a consequence, for expressions of the form XTY. Several comparisons show that the resulting estimates are, in general, not comparable with classical bounds, but can be sharper in specific cases.

Reviewer: Frédéric Morneau-Guérin (Québec)

MSC:

15A45 Miscellaneous inequalities involving matrices
15A60 Norms of matrices, numerical range, applications of functional analysis to matrix theory
15A18 Eigenvalues, singular values, and eigenvectors
47A12 Numerical range, numerical radius
47A30 Norms (inequalities, more than one norm, etc.) of linear operators
47A63 Linear operator inequalities

Keywords:

Cauchy-Schwarz inequality; singular value; operator norm; numerical radius

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