A question of Erdős on doubly stochastic matrices

Ludovick Bouthat

Université Laval

CanaDAM

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The permanent function

Definition

Let $A \in M_n$. The *permanent* of A is :

$$\operatorname{perm}(A) := \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)}.$$

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- This is reminiscent to the *determinant* :

$$\det(A) := \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}.$$

Facts about the permanent

- If A is an $n \times n$ $(0,1)$-matrix, then $\text{perm}(A)$ is the number of permutation matrices P with $P \leq A$ entrywise.

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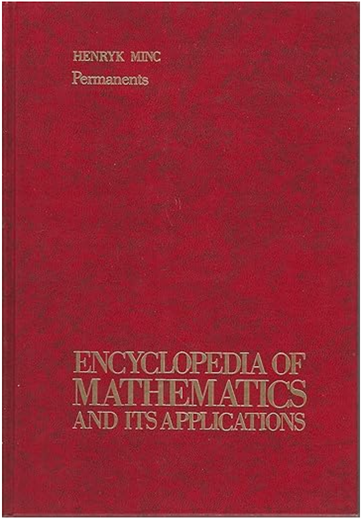
- 201 papers published between 1970–1976.
- 362 papers published between 1976–1982.

A book on permanents!



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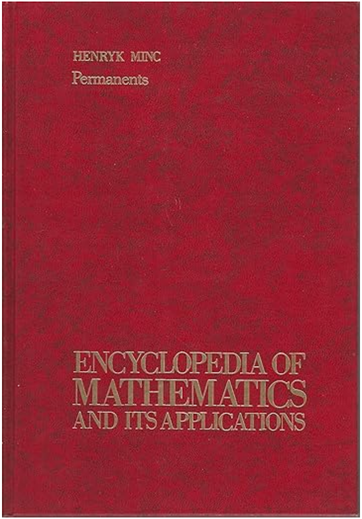

van der Waerden's conjecture (1926)

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If A is an $n \times n$ doubly stochastic matrix, then

$$\mathrm{perm}(A) \geq \frac{n!}{n^n},$$

with equality if and only if $A = J_n$, the matrix whose entries are all $\frac{1}{n}$.

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- Recall that a square matrix is *doubly stochastic* if :
  1. Its entries are nonnegative.
  2. The sum of the entries in each row is equal to 1.
  3. The sum of the entries in each column is equal to 1.

The start of a race (1959)

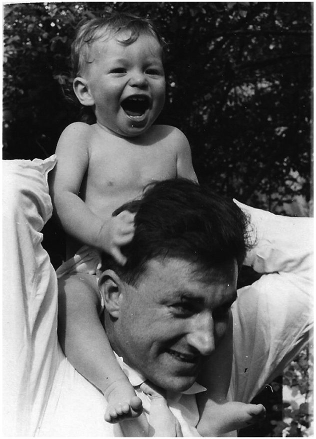


Figure: Marvin Marcus

Figure: Rimhak Ree

The end of a race (1981)

Figure: Georgy P. Egorychev

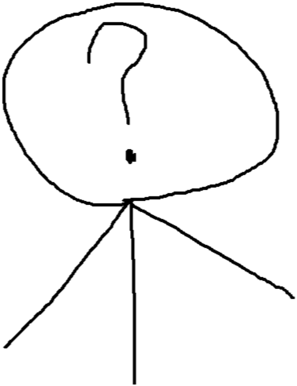
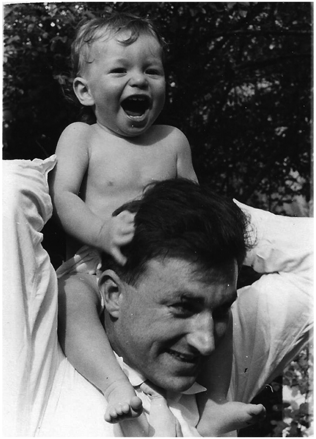


Figure: Dmitry I. Falikman

The start of a race (1959)



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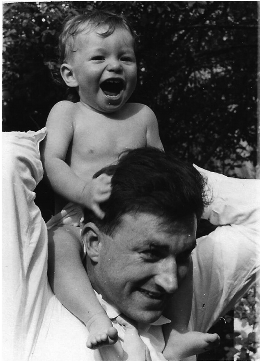

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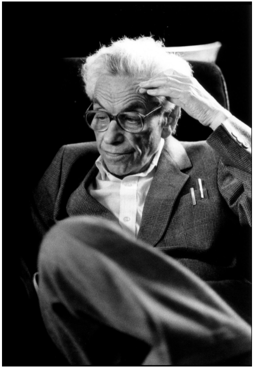

Figure: Paul Erdős

An interesting result

Theorem (Marcus–Ree, 1959)

If A is an $n \times n$ doubly stochastic matrix, then there exists a permutation $\sigma \in S_n$ such that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2 \leq \sum_{i=1}^{n} a_{i,\sigma(i)}.$$

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- If

$$A = \begin{bmatrix} 1/3 & 1/2 & 1/6 \\ 1/4 & 1/4 & 1/2 \\ 5/12 & 1/4 & 1/3 \end{bmatrix},$$

then $\frac{10}{9} = \sum_{i,j=1}^{n} a_{ij}^2 \leq \sum_{i=1}^{n} a_{i,\sigma(i)} = \frac{17}{12}$.

Some observations

Corollary (Marcus–Ree, 1959)

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- In other words,

$$\|A\|_{\mathrm{F}}^2 \le \max_{P \in \mathcal{P}_n} \operatorname{tr}(AP),$$

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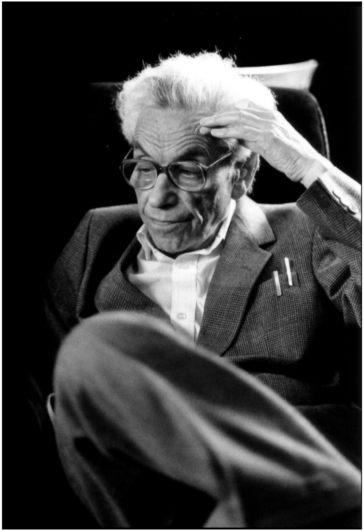
LHS = The Frobenius norm of A squared;
RHS = The *maximal trace* of A (related to the assignment problem).

Erdős' question

$$\text{When is } \|A\|_{\mathrm{F}}^2 = \max_{P \in \mathcal{P}_n} \operatorname{tr}(AP)\,?$$



More observations

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$$A = \begin{bmatrix} t & 1-t \\ 1-t & t \end{bmatrix}, \qquad 0 \leq t \leq 1.$$

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- $\|A\|_{\mathsf{F}}$ and $\max_{P \in \mathcal{P}_n} \operatorname{tr}(AP)$ are both invariant by left and right multiplication by a permutation matrix.

$\implies$ If A is an Erdős matrix, then so is PAQ for all $P, Q \in \mathcal{P}_n$.

- Similarly, if A is an Erdős matrix, then so is A^{tr}.

An elegant result

Proposition (Marcus–Ree, 1959)

If $A = P(J_{n_1} \oplus \cdots \oplus J_{n_r})Q$*, where* P *and* Q *are permutation matrices and* $n_1 + \cdots + n_r = n$*, then* A *is an Erdős matrix.*

A natural conjecture

Conjecture

A doubly stochastic matrix A satisfy $\|A\|_{\mathrm{F}}^2 = \max_{P \in \mathcal{P}_n} \operatorname{tr}(AP)$ if and only if $A = P(J_{n_1} \oplus \cdots \oplus J_{n_r})Q$, where P and Q are permutation matrices and $n_1 + \cdots + n_r = n$.

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- **Counter-example :**

$$A = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 1/4 & 1/4 \\ 1/2 & 1/4 & 1/4 \end{bmatrix}$$

is not of the conjectured form, but $\|A\|_F^2 = \frac{5}{4} = \max_{P\in\mathcal{P}_n} \operatorname{tr}(AP)$.

A second conjecture

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Every doubly stochastic matrix A satisfying $\|A\|_{\mathsf{F}}^2 = \max_{P \in \mathcal{P}_n} \operatorname{tr}(AP)$ can be turned into a symmetric matrix by permuting its rows and columns.

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- **Counter-example :**

$$A = \frac{1}{6}\begin{bmatrix} 3 & 3 & 0 & 0 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \end{bmatrix}$$

satisfy $\|A\|_F^2 = \dfrac{4}{3} = \max\limits_{P\in\mathcal{P}_n} \operatorname{tr}(AP)$ but cannot be turned into a symmetric matrix.

The case $n = 3$ (weak form)

Lemme (B.–Mashreghi–Morneau-Guérin, 2024)

Let A be a 3×3 doubly stochastic matrix. Then there is a permutation σ such that

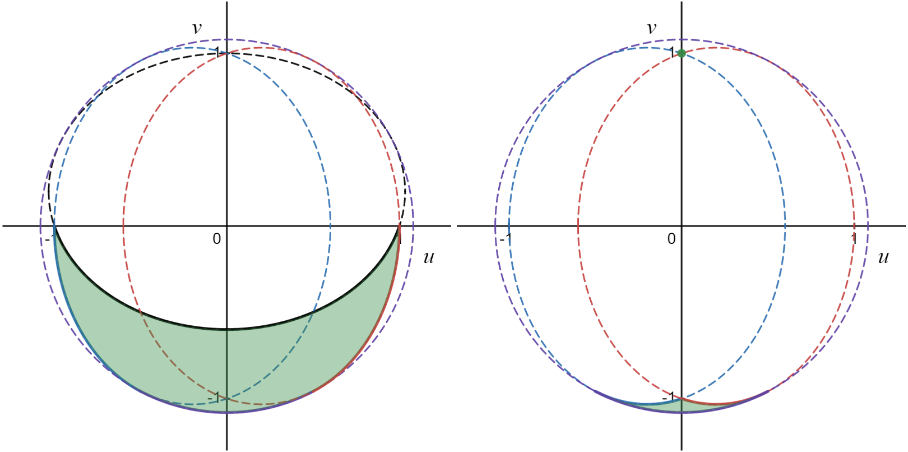
$$\|A\|_{\mathrm{F}}^2 = \sum_{i=1}^{3} a_{i,\sigma(i)}$$

if and only if either $A = J_3$, or

$$PAQ = \begin{bmatrix} \frac{v+u+3}{4} & \frac{1-2v\pm\sqrt{7-6u^2-6v^2}}{8} & * \\ 0 & \frac{v-u+3}{4} & * \\ * & * & * \end{bmatrix}$$

for some permutation matrices P and Q and parameters $(u, v) \in \mathcal{U}_{\pm}$.

The regions $\mathcal{U}_-$ and $\mathcal{U}_+$



(a) The region $\mathcal{U}_-$.

(b) The region $\mathcal{U}_+$.

The case $n = 3$ (stronger case)

Theorem (B.–Mashreghi–Morneau-Guérin, 2024)

Let A be a 3×3 doubly stochastic matrix. Then

$$\|A\|_F^2 = \max_{P \in \mathcal{P}_n} \operatorname{tr}(AP)$$

if and only if there are permutation matrices P and Q such that PAQ is equal to any of the following matrices:

(i) $R = \begin{bmatrix} \frac{3}{5} & 0 & \frac{2}{5} \\ 0 & \frac{3}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{2}{5} & \frac{1}{5} \end{bmatrix}$; (iii) $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; (v) $S = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$;

(ii) $T = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$; (iv) $J_3 = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$; (vi) $I_1 \oplus J_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$.

The case $n = 4$

Theorem (Tripathi–Kushwaha, 2025)

There are exactly 41 (up to a permutation of the rows and columns) 4×4 *Erdős matrices. Further, if we identify matrices up to transposition, then there are 32 unique* 4×4 *Erdős matrices.*

General case

Theorem (Tripathi, 2024)

The doubly stochastic matrices A satisfying $\|A\|_{\mathrm{F}}^2 = \max_{P \in \mathcal{P}_n} \mathrm{tr}(AP)$ *have rational entries.*

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Theorem (Tripathi, 2024)

Let $n \geq 4$. There are only finitely many Erdős matrices. More precisely,

$$\#\ \text{of Erdős matrices} \leq \sum_{j=1}^{(n-1)^2+1} \binom{n!}{j}.$$

References

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