Hardy-type Inequalities for ℓ^p sequences

Ludovick Bouthat

Université Laval

Operator analysis on function spaces; June 2025

Acknowledgement

This research is a collaborate effort with Pr. Javad Mashreghi and Pr. Frédéric Morneau-Guérin.

It was done with the financial help of the Vanier Scholarship.

Bourses d'études
supérieures du Canada
Vanier
Canada Graduate
Scholarships

Motivation

Weighted Dirichlet spaces

Definition

Let $\mathbb{D}$ be the open unit disk and $\mathrm{Hol}(\mathbb{D})$ be the set of holomorphic functions on $\mathbb{D}$. Let ω be a positive superharmonic function on $\mathbb{D}$ and $f \in \mathrm{Hol}(\mathbb{D})$. We define

$$\mathcal{D}_\omega(f) := \int_{\mathbb{D}} |f'(z)|^2 \omega(z) \mathrm{d}A(z),$$

where $\mathrm{d}A$ denotes the normalized area measure on $\mathbb{D}$. The *weighted Dirichlet space* $\mathcal{D}_\omega$ is the set of functions $f \in \mathrm{Hol}(\mathbb{D})$ such that $\mathcal{D}_\omega(f) < \infty$.

Motivation

Hadamard multipliers

Definition

Let $f(z) := \sum_{k=0}^{\infty} a_k z^k$ and $g(z) := \sum_{k=0}^{\infty} b_k z^k$ be two formal power series. Their *Hadamard product* is defined to be

$$(f * g)(z) := \sum_{k=0}^{\infty} a_k b_k z^k.$$

The *Hadamard mulipliers* of $\mathcal{D}_\omega$ are the formal power series h that have the property that $h * f \in \mathcal{D}_\omega$ for each $f \in \mathcal{D}_\omega$.

Motivation

Characterization of the Hadamard multipliers of $\mathcal{D}_\omega$

Theorem (Mashreghi, Ransford ; 2019)

Let $h(z)$ be a formal power series. The following statements are equivalent.
(i) h is an Hadamard multiplier of $\mathcal{D}_\omega$ for every superharmonic weight ω.
(ii) The L-matrix $[h_i - h_{i+1}]$ acts as a bounded operator on ℓ^2.

Let (a_n) be a sequence of complex number. An L-matrix is an infinite matrix of the form

$$A := [a_n] = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_1 & a_1 & a_2 & a_3 & \cdots \\ a_2 & a_2 & a_2 & a_3 & \cdots \\ a_3 & a_3 & a_3 & a_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

A partial characterization

Theorem (B., Mashreghi ; 2019)

Let $A = [a_n]$ be a positive L-matrix. The condition $a_n = O\left(1/n^{\alpha}\right)$ is

- *necessary if $\alpha = \frac{1}{2}$;*
- *nor necessary, nor sufficient if $\frac{1}{2} < \alpha < 1$;*
- *sufficient if $\alpha = 1$;*

for A to act as a bounded operator on ℓ^2.

A partial characterization

Theorem (B., Mashreghi ; 2019)

Let $A = [a_n]$ be a positive L-matrix. The condition $a_n = O(1/n^\alpha)$ is

- *necessary if $\alpha = \frac{1}{2}$;*
- *nor necessary, nor sufficient if $\frac{1}{2} < \alpha < 1$;*
- *sufficient if $\alpha = 1$;*

for A to act as a bounded operator on ℓ^2.

A partial characterization

Theorem (B., Mashreghi ; 2019)

Let $A = [a_n]$ be a positive L-matrix. The condition $a_n = O(1/n^{\alpha})$ is

- *necessary if $\alpha = \frac{1}{2}$;*
- *nor necessary, nor sufficient if $\frac{1}{2} < \alpha < 1$;*
- *sufficient if $\alpha = 1$;*

for A to act as a bounded operator on ℓ^2.

Consider the sequence $a_{4^n} = \frac{1}{2^n}$, and 0 otherwise. We show that

$$\sum_{n=1}^{\infty} \frac{1}{4^n} \left| \sum_{k=1}^{4^n} a_k \right|^2 \leq C \sum_{n=1}^{\infty} |a_n|^2.$$

Hardy's inequality

Theorem (Hardy, Riesz ; 1920)

If (a_n) is a sequence of positive numbers and $p \in (1, \infty)$, then

$$\sum_{n=1}^{\infty} \left(\frac{a_1 + a_2 + \cdots + a_n}{n} \right)^p \leq \left(\frac{p^2}{p-1} \right)^p \sum_{n=1}^{\infty} a_n^p.$$

Hardy's inequality

Theorem (Hardy, Riesz ; 1920)

If (a_n) is a sequence of positive numbers and $p \in (1, \infty)$, then

$$\sum_{n=1}^{\infty} \left(\frac{a_1 + a_2 + \cdots + a_n}{n} \right)^p \leq \left(\frac{p^2}{p-1} \right)^p \sum_{n=1}^{\infty} a_n^p .$$

Theorem (E. Landau ; 1926)

If (a_n) is a sequence of positive numbers and $p \in (1, \infty)$, then

$$\sum_{n=1}^{\infty} \left(\frac{a_1 + a_2 + \cdots + a_n}{n} \right)^p \leq \left(\frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} a_n^p .$$

Moreover, the constant $\left(\frac{p}{p-1}\right)^p$ is optimal.

A direct comparison

Hardy's inequality ($p = 2$)

$$\sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} a_k \right|^2 \leq C \sum_{n=1}^{\infty} |a_n|^2.$$

Our inequality

$$\sum_{n=1}^{\infty} \frac{1}{4^n} \left| \sum_{k=1}^{4^n} a_k \right|^2 \leq C \sum_{n=1}^{\infty} |a_n|^2.$$

The main goal

Hardy's inequality

$$\sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} 1 \cdot a_k \right|^p \leq C \sum_{n=1}^{\infty} |a_n|^p.$$

The main goal

Our inequality

$$\sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k \in \mathcal{N}_n} 1 \cdot a_k \right|^p \leq C \sum_{n=1}^{\infty} |a_n|^p.$$

$\mathcal{N}_n$: Indices sets.

The main goal

Our inequality

$$\sum_{n=1}^{\infty}\left|\frac{1}{n}\sum_{k\in\mathcal{N}_n} m_k a_k\right|^p \leq C\sum_{n=1}^{\infty}|a_n|^p.$$

$\mathcal{N}_n$: Indices sets.

m_k : Individual weights of the sequence a_n.

The main goal

Our inequality

$$\sum_{n=1}^{\infty} \left| \frac{1}{M_n} \sum_{k \in \mathcal{N}_n} m_k a_k \right|^p \leq C \sum_{n=1}^{\infty} |a_n|^p.$$

$\mathcal{N}_n$: Indices sets.

m_k : Individual weights of the sequence (a_n).

M_n : Overall weights of the sequence (a_n).

Definitions

Summation indices

- Let $\mathbb{N} = N_1 \cup N_2 \cup \cdots$ be a *block* partition of $\mathbb{N}$.

Definitions

Summation indices

- Let $\mathbb{N} = N_1 \cup N_2 \cup \cdots$ be a *block* partition of $\mathbb{N}$.
- Let $\mathcal{N}_n := N_1 \cup N_2 \cup \cdots \cup N_n$.

Definitions

Summation indices

- Let $\mathbb{N} = N_1 \cup N_2 \cup \cdots$ be a *block* partition of $\mathbb{N}$.
- Let $\mathcal{N}_n := N_1 \cup N_2 \cup \cdots \cup N_n$.

Observation : $\mathcal{N}_1 \subsetneq \mathcal{N}_2 \subsetneq \mathcal{N}_3 \subsetneq \cdots$

Definitions

Weights

• $\mathbb{N} = N_1 \cup N_2 \cup \cdots$ • $\mathcal{N}_n := N_1 \cup N_2 \cup \cdots \cup N_n$

- $(m_n)_{n\geq 1}$ is a sequence of positive numbers (individual weights).

Definitions

Weights

• $\mathbb{N} = N_1 \cup N_2 \cup \cdots$ • $\mathcal{N}_n := N_1 \cup N_2 \cup \cdots \cup N_n$

- $(m_n)_{n\geq 1}$ is a sequence of positive numbers (individual weights).
- $w_n := \Big(\sum_{k\in N_n} m_k^q\Big)^{1/q}$ (block weights).

Definitions

Weights

• $\mathbb{N} = N_1 \cup N_2 \cup \cdots$ • $\mathcal{N}_n := N_1 \cup N_2 \cup \cdots \cup N_n$

- $(m_n)_{n \geq 1}$ is a sequence of positive numbers (individual weights).
- $w_n := \left(\sum_{k \in N_n} m_k^q \right)^{1/q}$ (block weights).
- $M_n := \sum_{k=1}^{n} w_k$ (overall weights).

Definitions

Weights

• $\mathbb{N} = N_1 \cup N_2 \cup \cdots$ • $\mathcal{N}_n := N_1 \cup N_2 \cup \cdots \cup N_n$

- $(m_n)_{n\geq 1}$ is a sequence of positive numbers (individual weights).
- $w_n := \left(\sum_{k\in N_n} m_k^q\right)^{1/q}$ (block weights).
- $M_n := \sum_{k=1}^{n} w_k$ (overall weights).
- $\rho := \sup_{n\geq 1}\left(w_n \sum_{k=n}^{\infty} \frac{1}{M_k}\right).$

Main result

- $\mathbb{N} = N_1 \cup N_2 \cup \cdots$
- $\mathcal{N}_n := N_1 \cup \cdots \cup N_n$
- (m_n) is a sequence of positive numbers
- $w_n := \left(\sum_{k \in N_n} m_k^q\right)^{1/q}$
- $M_n := \sum_{k=1}^{n} w_k$
- $\rho := \sup_{n \geq 1} \left(w_n \sum_{k \geq n} 1/M_k \right)$

Main result

- $\mathbb{N} = N_1 \cup N_2 \cup \cdots$
- $\mathcal{N}_n := N_1 \cup \cdots \cup N_n$
- (m_n) is a sequence of positive numbers
- $w_n := \left(\sum_{k \in N_n} m_k^q\right)^{1/q}$
- $M_n := \sum_{k=1}^{n} w_k$
- $\rho := \sup_{n \geq 1} \left(w_n \sum_{k \geq n} 1/M_k \right)$

Theorem (B., Mashreghi, Morneau-Guérin ; 2023)

Let $(a_n)_{n\geq 1}$ be a sequence of complex numbers, and let $(m_n)_{n\geq 1}$ be a sequence of weights. Define M_n and ρ as above, and assume that ρ is finite. Then, for $p > 1$,

$$\sum_{n=1}^{\infty} \left| \frac{1}{M_n} \sum_{k \in \mathcal{N}_n} m_k a_k \right|^p \leq \rho \sum_{n=1}^{\infty} |a_n|^p.$$

Main result

- $\mathbb{N} = N_1 \cup N_2 \cup \cdots$
- $w_n := \left(\sum_{k \in N_n} m_k^q\right)^{1/q}$
- $\mathcal{N}_n := N_1 \cup \cdots \cup N_n$
- ~~$M_n := \sum_{k=1}^n w_k$~~
- (m_n) is a sequence of positive numbers
- ~~$\rho := \sup_{n\geq 1} \left(w_n \sum_{k\geq n} 1/M_k\right)$~~

Theorem (B., Mashreghi, Morneau-Guérin ; 2023)

Let $(a_n)_{n\geq 1}$ be a sequence of complex numbers, and let $(m_n)_{n\geq 1}$ be a sequence of weights. Define M_n and ρ as above, and assume that ρ is finite. Then, for $p > 1$,

$$\sum_{n=1}^{\infty} \left| \frac{1}{M_n} \sum_{k \in \mathcal{N}_n} m_k a_k \right|^p \leq \rho \sum_{n=1}^{\infty} |a_n|^p.$$

Main result

- $\mathbb{N} = N_1 \cup N_2 \cup \cdots$
- $\mathcal{N}_n := N_1 \cup \cdots \cup N_n$
- (m_n) is a sequence of positive numbers
- $w_n := \left(\sum_{k\in N_n} m_k^q\right)^{1/q}$
- ~~$M_n := \sum_{k=1}^n w_k$~~
- ~~$\rho := \sup_{n\geq 1}\left(w_n \sum_{k\geq n} 1/M_k\right)$~~

Theorem (B., Mashreghi, Morneau-Guérin ; 2023)

Let $(a_n)_{n\geq 1}$ be a sequence of complex numbers, and let $(m_n)_{n\geq 1}$ and $(M_n)_{n\geq 1}$ be two sequences of weights. If $p > 1$ and

$$\rho := \sup_{n\geq 1}\left(w_n \sum_{k=n}^{\infty} \frac{(w_1 + \cdots + w_k)^{p-1}}{M_k^p}\right)$$

is finite, then

$$\sum_{n=1}^{\infty}\left|\frac{1}{M_n}\sum_{k\in\mathcal{N}_n} m_k a_k\right|^p \leq \rho \sum_{n=1}^{\infty} |a_n|^p.$$

Example

- $\mathbb{N} = N_1 \cup N_2 \cup \cdots$
- $w_n := \left(\sum_{k \in N_n} m_k^q\right)^{1/q}$
- $\mathcal{N}_n := N_1 \cup \cdots \cup N_n$
- ~~$M_n := \sum_{k=1}^n w_k$~~
- (m_n) is a sequence of positive numbers
- ~~$\rho := \sup_{n \geq 1} \left(w_n \sum_{k \geq n} 1/M_k\right)$~~

Example

If $N_n = \{n\}$, $m_n = 1$ and $M_n = n^{1+\varepsilon}$ $(\varepsilon > 0)$ for all $n \geq 1$, then

- $\mathcal{N}_n = \{1, 2, \ldots, n\}$;
- $w_n = 1$;
- $\rho = \sup_{n \geq 1} \sum_{k=n}^{\infty} \frac{1}{k^{1+p\varepsilon}} \leq \zeta(1 + p\varepsilon) < \infty$.

Example

- $\mathbb{N} = N_1 \cup N_2 \cup \cdots$
- $w_n := \left(\sum_{k \in \mathcal{N}_n} m_k^q\right)^{1/q}$
- $\mathcal{N}_n := N_1 \cup \cdots \cup N_n$
- ~~$M_n := \sum_{k=1}^n w_k$~~
- (m_n) is a sequence of positive numbers
- ~~$\rho := \sup_{n \geq 1} \left(w_n \sum_{k \geq n} 1/M_k\right)$~~

Example

If $N_n = \{n\}$, $m_n = 1$ and $M_n = n^{1+\varepsilon}$ $(\varepsilon > 0)$ for all $n \geq 1$, then

- $\mathcal{N}_n = \{1, 2, \ldots, n\}$;
- $w_n = 1$;
- $\rho = \sup_{n \geq 1} \sum_{k=n}^{\infty} \frac{1}{k^{1+p\varepsilon}} \leq \zeta(1 + p\varepsilon) < \infty$.

$$\implies \sum_{n=1}^{\infty} \left| \frac{a_1 + \cdots + a_n}{n^{1+\varepsilon}} \right|^p \leq \zeta(1 + p\varepsilon) \sum_{n=1}^{\infty} |a_n|^p.$$

Corollaries

An application to lacunary sequences

Corollary (B., Mashreghi, Morneau-Guérin ; 2023)

Let $(a_n)_{n\geq 1}$ be a sequence of complex numbers, and let $(n_k)_{k\geq 1}$ be a lacunary sequence satisfying $\frac{n_{k+1}}{n_k} \geq r$ $(k \geq 1)$. If $p > 1$, then

$$\sum_{k=1}^{\infty} \left| \frac{1}{n_k^{1/q}} \sum_{j=1}^{n_k} a_j \right|^p \leq \left(\frac{r^{1/q}}{r^{1/q} - 1} \right)^p \sum_{n=1}^{\infty} |a_n|^p.$$

Corollaries

An extreme case : Geometric sequences

Theorem (B., Mashreghi, Morneau-Guérin ; 2023)

Let $(a_n)_{n\geq 1}$ be a sequence of complex numbers and let $b \geq 2$ be an integer. Then

$$\sum_{k=1}^{\infty} \frac{1}{b^k} \left| \sum_{j=1}^{b^k} a_j \right|^2 \leq \frac{\sqrt{b}+1}{\sqrt{b}-1} \sum_{n=1}^{\infty} |a_n|^2.$$

Moreover, the constant $\frac{\sqrt{b}+1}{\sqrt{b}-1}$ is optimal and the above inequality is strict, except if $(a_n)_{n\geq 1}$ is the null sequence.

Corollaries

An extreme case : Geometric sequences

Example

If $b = 4$, then

$$\sum_{n=1}^{\infty} \frac{1}{4^n} \left| \sum_{k=1}^{4^n} a_k \right|^2 \leq 3 \sum_{n=1}^{\infty} |a_n|^2.$$

Moreover, the constant 3 is optimal.

Generalization to functions spaces

Theorem (Vincent, Sohani ; 2025+)

Let $A_1, A_2, \ldots$ be a partition of $\mathbb{R}^m$ such that each A_i is measurable. Let $B_n = \bigcup_{i=1}^n A_i$ and let $p > 1$. For $g > 0$ defined on $\mathbb{R}^m$, let

$$w_n = \left(\int_{A_n} g(x)^q \, \mathrm{d}x \right)^{1/q}.$$

If M_n is a sequence of positive numbers such that

$$\rho := \sup_k \sum_{n=k}^{\infty} \frac{w_k \left(\sum_{j=1}^n w_j \right)^{p/q}}{M_n^p} < \infty,$$

then

$$\sum_{n=1}^{\infty} \left| \frac{1}{M_n} \int_{B_n} g(x) f(x) \, \mathrm{d}x \right|^p \leq \rho \int_{\mathbb{R}^m} |f(x)|^p \, \mathrm{d}x.$$

Generalization to functions spaces

Corollary (Vincent, Sohani ; 2025+)

For $k \geq 0$,

$$\int_0^\infty x^{-r}\left(\int_0^{kx} f(t)\,\mathrm{d}t\right)^p \mathrm{d}x \leq k^{r-1}\left(\frac{p}{r-1}\right)^p \int_0^\infty f(x)^p x^{p-r}\,\mathrm{d}x,$$

in which the constant is optimal.

References

Ludovick Bouthat & Javad Mashreghi (2021) L-matrices with lacunary coefficients, *Operators and Matrices*, 15(3) :1045–1053.

Ludovick Bouthat & Javad Mashreghi (2021) The norm of an infinite L-matrix, *Operators and Matrices*, 15(1) :47–58.

Ludovick Bouthat, Javad Mashreghi & Frédéric Morneau-Guérin (2023) Hardy-type inequalities for ℓ^p sequences. *Journal of Mathematical Inequalities.*, 17(3) :1165–1178.

Sajin Vincent A.W. & Vijay Kumar Sohani (2025+) Generalized Hardy Inequalities with Nonlinear Integration Limits. *To appear*.