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On the rational generating functions of Toeplitz matrices. (English) Zbl 08143737

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In their 2010 paper [Oper. Theory: Adv. Appl. 199, 25–118 (2010; Zbl 1203.15020)] *G. Heinig* and *K. Rost*, introduced the rational generating functions of an $n \times n$ Toeplitz matrix and presented a parameterized formula for the rational generating functions with McMillan degree n when the Toeplitz matrix is nonsingular. Building on this pioneering work, the authors of the article under consideration here study the rational generating functions of an arbitrary nonzero Toeplitz matrix (not necessarily square) with a prescribed allowable McMillan degree.

Denote the monic greatest common divisor of a pair of complex polynomials $s(z)$ and $t(z)$ by $\gcd\big(s(z), t(z)\big)$. If $r(z)$ is a nonzero rational function, then there exists a pair of nonzero polynomials $s(z)$ and $t(z)$ such that

$$f(z) = \frac{s(z)}{t(z)}, \quad \gcd\big(s(z), t(z)\big) = 1.$$

The McMillan degree of the rational function $r(z)$ is defined by

$$\deg r(z) := \max\big\{\deg s(z)\,,\ \deg t(z)\big\}.$$

Given a nonzero Toeplitz matrix $T := (a_{i-j})_{i,j=0}^{m-1,n-1} \in \mathbb{C}^{m\times n}$ with $a_0 = a_0^+ + a_0^-$, the rational generating function problem can be formulated in the following manner: find all rational functions $\varphi(z)$ with prescribed mcMillan degree, whose power series expansions at the origin and at infinity are

$$\varphi(z) = a_0^+ + a_1 z + a_2 z^2 + \cdots + a_{m-1} z^{m-1} + \mathcal{O}(z^m) \quad (z \to 0)$$

and

$$\varphi(z) = -a_0^- - \frac{a_{-1}}{z} - \frac{a_{-2}}{z^2} - \cdots - \frac{a_{1-n}}{z^{n-1}} + \mathcal{O}(z^{-n}) \quad (z \to \infty)$$

respectively. If a rational function $\varphi(z)$ with $\deg \varphi(z) = r$ both of the above, then $\varphi(z)$ is referred to as a rational generating function of T with McMillan degree r. In this case, r is referred to as an allowable McMillan degree of T.

Two questions naturally arise here:

1. Which positive integer is an allowable McMillan degree of T ?
2. How to describe the rational generating functions with a prescribed allowable McMillan degree ?

In the present paper, the authors provide complete answers to these two questions in the case where T is a nonzero Toeplitz matrix by introducing the *characteristic degrees* and the *characteristic polynomial pair of* T. It is important to emphasize that the characteristic degrees defined here are different from the characteristics of a nonzero Toeplitz matrix as defined and used by Heinig and Rost.

At the end of the paper, the authors present two examples to demonstrate the feasibility and efficiency of their proposed method.

Reviewer: Frédéric Morneau-Guérin (Québec)

MSC:

15B05 Toeplitz, Cauchy, and related matrices
30C10 Polynomials and rational functions of one complex variable
41A20 Approximation by rational functions

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Toeplitz matrix; rational generating function; McMillan degree; characteristic degree; characteristic polynomial pair

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