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Probabilistic zero forcing with vertex reversion. (English) Zbl 08113571

Electron. J. Linear Algebra 41, 49-73 (2025).

Suppose a graph is colored so that every vertex is either blue or white. The (deterministic) zero forcing color-change rule describes how blue vertices infect neighboring white vertices: a blue vertex u forces (changes) a white neighbor w to blue if w is its unique white neighbor.

This coloring process was introduced independently as a tool for controlling quantum systems and as a bound on the maximum nullity of a matrix in the study of the minimum rank problem. Since then, zero forcing has been found to have links with graph-search algorithms, power domination, and the Cops and Robbers game. It has thus become a topic of interest in its own right, giving rise to several variants.

One such variant is probabilistic zero forcing (PZF), first introduced by *C. X. Kang* and *E. Yi* [Bull. Inst. Comb. Appl. 67, 9–16 (2013; Zbl 1274.05475)] and forming the foundation of the present paper. Building on PZF, the author introduces reversion probabilistic zero forcing (RPZF), a modification in which blue vertices may revert to white at the end of each round. Two versions of this process are defined:

1. Single absorption reversion probabilistic zero forcing (SARPZF). In each round, phase 1 consists of each blue vertex independently attempting to force each of its white neighbors, with success governed by a fixed forcing probability; in phase 2, each blue vertex independently attempts to revert to white, again according to a prescribed reversion probability.
2. Dual absorption reversion probabilistic zero forcing (DARPZF). This variant modifies the SARPZF rule by adding one restriction: after the probabilistic forcing attempts of phase 1, if all vertices have become blue, then no probabilistic reversion occurs in phase 2 – in other words, the reversion step is suppressed whenever the graph becomes fully blue.

The main results of the paper describe the behavior of RPZF on complete and complete bipartite graphs under various initial densities of infected vertices. Quantities such as the threshold number of initially infected vertices required to fully infect the graph in a single time step, as well as asymptotic behavior, are analyzed. It is also shown that the star graph is comparatively more difficult to fully infect than complete or balanced complete bipartite graphs, a notion explored further through simulations.

The paper also provides standard Markov-chain results for RPZF processes and associated parameters on arbitrary graphs.

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MSC:

15B51	Stochastic matrices
60J10	Markov chains (discrete-time Markov processes on discrete state spaces)
60G50	Sums of independent random variables; random walks
05C15	Coloring of graphs and hypergraphs
05C81	Random walks on graphs
60J20	Applications of Markov chains and discrete-time Markov processes on general state spaces (social mobility, learning theory, industrial processes, etc.)
60J22	Computational methods in Markov chains
60C05	Combinatorial probability

Keywords:

probabilistic zero forcing; Markov chains on graphs; reversion; discrete contact process; high probability

Full Text: DOI arXiv

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