Moslehian, Mohammad Sal; Osaka, Hiroyuki
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Frontiers in Mathematics. Cham: Birkhäuser (ISBN 978-3-031-64545-7/pbk; 978-3-031-64546-4/ebook). x, 218 p. (2024).

Advanced Techniques with Block Matrices of Operators by Mohammad Sal Moslehian and Hiroyuki Osaka is a research-level monograph that offers an excellent, modern, and relatively self-contained resource for readers with a solid background in operator theory. The opening chapter provides a dense yet concise refresher on foundational tools from functional analysis, operator theory, and matrix analysis. While it briefly introduces elementary notions (such as norms and Hilbert spaces) it quickly moves on to more advanced concepts like the Gelfand-Beurling formula and a sketch of the Gelfand-Naimark-Segal representation. This chapter is not intended as a textbook for introductory courses; rather, it presupposes substantial familiarity with the rudiments of functional analysis and operator theory. It continues with key results such as Douglas' majorization theorem and the polar decomposition, followed by three useful digressions: one on unitarily invariant norms, one on the Moore-Penrose inverse, and one clarifying the distinction between real and complex Hilbert spaces.

The second chapter offers a clear and rigorous presentation of block operator matrices. Foundational results are provided in full detail, with minimal recourse to external references or exercises to complete essential proofs. This chapter forms one of the two conceptual pillars of the book. Classical techniques such as Schur complements and dilation theory are developed and employed to derive major results including Sz.-Nagy's dilation theorem, the von Neumann inequality, Putnam-Fuglede-type theorems, Fiedler's inequality, and Hua's determinantal inequality.

Subsequent chapters continue this development by integrating tools from the theory of completely positive maps, and related operator inequalities. The authors also introduce advanced topics such as Lieb maps and 3-positivity using operator block techniques. Throughout, block matrix techniques serve as the main analytical machinery. The monograph also explores applications in quantum information theory, C^*-algebra methods, and various notions of operator positivity, making it particularly relevant to both mathematicians and physicists. It should be emphasized that the authors do not aim to produce an encyclopedic treatment. While the book delves deeply into operator inequalities and C^*-algebraic positivity, it offers limited coverage of spectral or numerical applications for large block structures or infinite-dimensional systems. Researchers focused on topics such as Schur complements for large block matrices or operator methods in boundary value problems will need to consult supplementary resources.

This work is unmistakably modern. Its bibliography includes close to 250 references, featuring both classical texts and a substantial number of contemporary sources, particularly from the first two decades of the 21st century. These references are especially strong in the areas of operator inequalities and the structure of C^*-algebras. Despite how it is marketed by the publisher, this monograph is far better suited to advanced graduate-level seminars or independent study by researchers than to inclusion in general undergraduate or first-year graduate curricula. Readers with a firm grounding in linear algebra and functional analysis will find it especially rewarding.

The book focuses primarily on 2×2 block operator matrices. As the authors note, even these restricted cases give rise to deep and nontrivial theory. Nevertheless, researchers working with more general block structures or advanced spectral questions will likely require further references. This does not, however, diminish the quality or relevance of the monograph. The substantial number of exercises, problems, and research questions (many anchored in the recent literature) underscore its depth and utility for those engaged in serious study or research in the field.

Reviewer: Frédéric Morneau-Guérin (Québec)

MSC:

47-02 Research exposition (monographs, survey articles) pertaining to operator theory
47A08 Operator matrices
47A63 Linear operator inequalities
47A64 Operator means involving linear operators, shorted linear operators, etc.
47A20 Dilations, extensions, compressions of linear operators

Keywords:

block matrices of operator; positive operator

Full Text: DOI