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Some observations on Erdős matrices. (English. English summary)
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A doubly stochastic matrix (also called bistochastic matrix) is a square matrix of nonnegative real numbers, each of whose rows and columns sums to 1.

In 1959, Marcus and Ree proved that any $n \times n$ doubly stochastic matrix A satisfies

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}^2 \leq \max_{\sigma \in S_n} \sum_{i=1}^{n} A_{i,\sigma(i)},$$

where S_n is the set of all permutations of the integers $1, 2, \ldots, n$. To put it another way, the Marcus-Ree inequality says that any doubly stochastic matrix has a maximal trace (or maximal diagonal sum) greater than or equal to its Frobenius norm squared.

Seeing this inequality, Erdős asked for which doubly stochastic matrices it is saturated. Such matrices will henceforth be referred to as Erdős matrices. Since both the functions

$$A \mapsto \|A\|_{\mathrm{F}}^2 := \sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}^2 \quad \text{and} \quad A \mapsto \operatorname{maxTrace}(A) := \max_{\sigma \in S_n} \sum_{i=1}^{n} A_{i,\sigma(i)}$$

remain unchanged if A is replaced by PAQ for some permutations matrices P and Q, the Erdős matrices must be considered up to a permutation.

While characterizing 2×2 Erdős matrices is trivial, it is only recently that a complete characterization (up to equivalence) of 3×3 Erdős matrices was obtained. The result—due to Bouthat, Mashreghi and Morneau-Gurin (2024)—appears to have revived interest in the question raised by Erdős.

Two new results were obtained in the article under evaluation.

First, it is shown that, for $n \geq 4$, there are (with some equivalence) a finite number of $n \times n$ Erdős matrices. If the demonstration provides an upper bound for the number of such matrices that is anything but sharp, it has the advantage of providing an algorithmic procedure for testing and generating Erdős matrices. By way of example, the author revisits the case of dimension $n = 3$. Although the results obtained in doing so are not original, the method used to obtain them has the advantage of being more revealing of the underlying dynamics.

Second, a question raised by Bouthat, Mashreghi and Morneau-Gurin is resolved. It is shown that every Erdős matrix has rational entries.

Finally, another interesting aspect of this article (which is independent of the other two) is that it proposes a natural generalization of Erdős' question. The author defines a real-valued function Δ_n on the set of all $n \times n$ doubly stochastic matrices as follows:

$$\Delta_n(A) := \max_{\sigma S_n} A_{i,\sigma_i} - \sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}^2.$$

The Marcus-Ree inequality is equivalent to $\Delta_n \geq 0$, and characterizing Erdős matrices (up to equivalence) amounts to characterizing the zero-set of Δ_n. For $n \geq 2$ and $\alpha \geq 0$, the author asks for which doubly stochastic matrices we have $\Delta_n = \alpha$. A complete solution is given in the case $n = 2$, and the author also reproduces a demonstration (obtained by Ottolini and Tripathi in a forthcoming paper) that $\Delta_n(A) \leq \frac{n-1}{4}$ for each $n \geq 2$.

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.