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On the submultiplicativity of matrix norms induced by random vectors. (English. English summary)

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In a recent two-part series of articles, A. A. Chavez, S. R. Garcia and J. Hurley [Canad. Math. Bull. **66** (2023), no. 3, 808–826; MR4651637; Canad. Math. Bull. **67** (2024), no. 2, 447–457; MR4751519] introduced a family of norms on the space M_n of $n \times n$ complex matrices that are induced by real-valued random vectors whose entries are independent and identically distributed (iid) random variables having finite d-moments where d satisfies $d \geq 2$. These norms arise from a probabilistic framework, and their construction and validation involve probability theory, partition combinatorics, and trace polynomials in noncommuting variables.

The first paper culminated in a series of open problems, one of which called for a characterization of those random vectors that give rise to submultiplicative norms. In the article under review, this question is partially answered by showing that for *any* real number $d \geq 1$ and *any* random variable $X \in L^{\max\{2+\varepsilon,d\}}(\Omega,\mathcal{F},\mathbb{P})$, there exists a positive constant γ_d, which is independent of n, such that the norm induced by X becomes submultiplicative when multiplied by γ_d.

In Section 2, before tackling the main question, the author shows that the family of norms defined by Chávez, Garcia and Hurley can be extended further to any real $d \geq 1$. Consequently, the question of Chávez, Garcia and Hurley is extended too.

Section 3 presents, for the sake of completeness, a number of classic inequalities (Jensen's, Lyapunov's, and a few others) that will be used in the proofs. In Section 4, a pair of useful lower and upper bounds for the norms at the heart of this paper are derived.

Section 5 contains the proof of the main result. The special case $d = 2$ is dealt with first, as it is then used as a basis for dealing with the general case. An enlightening example is presented in Section 6. Finally, three open problems (two of which are seeking generalizations of the family of norms and one which inquires about the possibility of loosening the conditions in the result of Section 5) are listed in Section 7.

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.