Singhal, Ritika; Kumar, N. Shravan
Paley inequality for the Weyl transform and its applications. (English) Zbl 07963422
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The Hausdorff-Young inequality is a foundational result in Fourier analysis which admits several generalizations.

The main aim of the paper under consideration is to study Paley's extension of the Hausdorff-Young inequality and to establish variants for the Weyl transform associated to locally compact abelian groups.

Following a presentation of the historical and conceptual context (Section 1), and a detailed presentation of the results on which the proofs of the main theorems are based (Section 2), two generalizations of the Hausdorff-Young theorem are obtained. The first extends the Hausdorff-Young theorem to Lorentz spaces while the second (a version of the Paley inequality) is a more generalized result that extends the Hausdorff-Young theorem to non-commutative Lorentz spaces on the Banach algebra of all bounded operators on $L^2(G)$ where G is a locally compact abelian group.

By interpolating the Paley inequality and the Hausdorff-Young inequality for the Weyl transform, some Hausdorff-Young Paley inequality is then obtained. This last result is then used to derive the Weyl transform analogue of the Hörmander's theorem.

Section 4 contains a demonstration of a Hausdorff-Young-Paley inequality for the *inverse* Weyl transform which is then put to use in deriving an analogue of the Hörmander's theorem for the *inverse* Weyl transform.

In Section 5, a study of the Hardy-Littlewood inequality is carried out. The scalar version is first proved as an application of the Paley inequality. A demonstration of the vector-valued case is then obtained.

The vector-valued version of the Paley inequality is obtained in Section 6. This is followed by a presentation of the concepts of Weyl-Paley type/cotype, Weyl type/cotype and Weyl-HL type/cotype, as well as a study of their duality.

In the last section of this paper, a version of Pitt's inequality (which can be seen as a generalized Hausdorff-Young inequality) of for the Weyl transform is derived as an application of Paley's inequality and some interpolation argument.

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MSC:

43A32 Other transforms and operators of Fourier type
42B15 Multipliers for harmonic analysis in several variables
43A15 L^p-spaces and other function spaces on groups, semigroups, etc.
43A25 Fourier and Fourier-Stieltjes transforms on locally compact and other abelian groups
43A40 Character groups and dual objects

Keywords:

Paley inequality; Hardy-Littlewood inequality; Weyl transform; Weyl multipliers; Hörmander's theorem; Pitt's inequality

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