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A face of the polytope of doubly stochastic matrices. (English. English summary)

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Let Ω_n be the Birkhoff polytope, i.e., the convex polytope of $n\times n$ doubly stochastic matrices. Given an $n\times n$ non-negative matrix $D=[d_{ij}]$, let

$$\Omega(D)=\Big\{X=[x_{ij}]\in\Omega_n:\ x_{ij}=0 \text{ whenever } d_{ij}=0\Big\}.$$

Then $\Omega(D)$ is a face of Ω_n, and compactness entails there exists a matrix minimizing the permanent on $\Omega(D)$. Recall that the *permanent* of an $n\times n$ matrix $A=[a_{i,j}]$ is defined by

$$\operatorname{per}(A)=\sum_\sigma a_{1,\sigma(1)}\cdots a_{n,\sigma(n)},$$

where σ runs over all permutations of $\{1,2,\dots,n\}$.

In this paper, the authors consider the face of Ω_n whose non-zero entries coincide with that of

$$V_{l,m,n}=\begin{pmatrix} 0_{l,l} & 0_{l,m} & J_{l,n}\\ 0_{m,l} & I_m & J_{m,n}\\ J_{n,l} & J_{n,m} & J_{n,n}\end{pmatrix},$$

where $0_{r,s}$ denotes the $r\times s$ zero matrix, $J_{u,v}$ is the $u\times v$ all-ones matrix, and I_m is the identity matrix of order m.

In Section 2, it is shown that the matrix minimizing the permanent on the faces $\Omega(V_{l,m,n})$ for $l\ge 1$, $m\ge 2$, $n\ge 2$, and $l<n$, is of the form

$$B(b)=\begin{pmatrix} 0_{l,l} & 0_{l,m} & \frac{1}{n}J_{l,n}\\ 0_{m,l} & dI_m & aJ_{m,n}\\ \frac{1}{n}J_{n,l} & aJ_{n,m} & bJ_{n,n}\end{pmatrix},$$

with $b\in[0,\frac{n-l}{n^2}]$ for $m+l\ge n$, and $b\in[\frac{1}{n^2}(n-l-m),\frac{n-l}{n^2}]$ for $m+l\le n$.

Moreover, it is shown through rather explicit calculations that

- $B(0)$ is the unique minimizing matrix in $\Omega(V_{l,m,n})$ if $m\ge 2(n-l)$;
- $B(0)$ is not a minimizing matrix in $\Omega(V_{l,m,n})$ if $(n-l)<m<(n-l)+\sqrt{n-l}$;
- $B(\frac{n-l}{n^2})$ is not a minimizing matrix in $\Omega(V_{l,m,n})$ if $l+m\ge n$.

This section also includes some results about ranges of values of l,m,n for which $V_{l,m,n}$ is a cohesive matrix, a barycentric matrix, or neither, thus contributing towards the solution of two problems from Minc's well-known lists of unsolved problems on permanents.

In Section 3, the authors investigate a special case where $(n-l)+\sqrt{n-l}<m<2(n-l)$, that is the case $l=2k$, $m=7k$, and $n=6k$ with $k\ge 2$. In this special case, it is shown that $B(0)$ is the unique minimizing matrix on $\Omega(V_{l,m,n})$.

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.