MR4758196 15B51 05A05 15A03 52A20

Cao, Lei (1-NOVA-M); **Bayati Eshkaftaki, Ali** (IR-SKUM-M);
Koyuncu, Selcuk (1-NGEOR2-M)

Variations in the sub-defect of doubly substochastic matrices. (English. English summary)

Spec. Matrices **12** (2024), *Paper No.* 20240012, 10 *pp.*

A square matrix of nonnegative real numbers is said to be *doubly stochastic* if the sum of the elements in each row and column is equal to 1 and *doubly substochastic* if the sum of the elements in each row and column is less than or equal to 1.

L. Cao, S. Koyuncu & T. Parmer (2016) introduced the concept of the *sub-defect* of an $n \times n$ doubly substochastic matrix B, denoted by $sd(B)$, which is defined as the minimum number of rows and columns needed to be added to B to obtain a doubly stochastic matrix A containing B as a submatrix. Moreover, they showed that the sub-defect of B can be easily calculated using the sum of all elements of B, i.e.,

$$sd(B) = \left\lceil n - \sum_{i=1}^{n} \sum_{j=1}^{n} b_{i,j} \right\rceil .$$

In the present paper, the authors explore sub-defects for four classes of doubly substochastic matrices each displaying a certain type of symmetry, namely (1) symmetric matrices (i.e., matrices that are equal to their transpose, obtained by reflecting the original matrix with respect to the main diagonal), (2) Hankel symmetric matrices (i.e., matrices that are equal to their Hankel transpose, obtained by reflecting the original matrix with respect to the main antidiagonal), (3) centrosymmetric matrices (i.e., matrices that are equal to the matrix obtained by rotating the original matrix by 180°), and (4) matrices that are simultaneously symmetric, Hankel symmetric and centrosymmetric.

In the first section of this article, which is mainly devoted to introducing the concepts at the heart of the article and presenting a brief review of the relevant literature, it is shown that the sets of all $n \times n$, symmetric doubly substochastic matrices, Hankel symmetric doubly substochastic matrices, centrosymmetric doubly substochastic matrices, and doubly substochastic matrices with all three of these symmetries are all compact and convex subsets of $\mathbb{R}^{n\times n}$.

In the second (and main) section of this article, the authors define four sub-defect concepts, each with respect to one of the four types of symmetry mentioned above. As an example, they define the Hankel symmetric sub-defect of B, denoted $sd_h(B)$, as the minimum number of rows and columns needed to be added to B to obtain a Hankel symmetric doubly stochastic matrix A containing B as a submatrix. Following these definitions, they show that if an $n \times n$ doubly substochastic matrix B has a symmetry, then the sub-defect with that symmetry is determined by its regular sub-defect, whereas if B is simply an $n \times n$ doubly substochastic matrix, then its sub-defect with a given symmetry is bounded below by $sd(B)$ and above by some function of n. The section ends with a few well-chosen illustrative examples.

In a brief third and final section, the authors exploit lower and upper bounds obtained by L. Cao and S. Koyuncu (2017) for sub-defects of the product of doubly substochastic matrices to obtain estimates for the (Hankel/centrosymmetric/all) symmetric sub-defects of the product of matrices satisfying certain assumptions.

Frédéric Morneau-Guérin