Bogoya, M.; Böttcher, A.; Grudsky, S. M.

Eigenvalues of Toeplitz matrices emerging from finite differences for certain ordinary differential operators. (English) Zbl 07960775

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This paper is devoted to the asymptotic behavior of individual eigenvalues of Hermitian Toeplitz matrices $T_n(a_n)$ of so-called symbols a_n emerging from finite linear combinations with non-negative coefficients of the differential operators $(-1)^k \frac{\mathrm{d}^{2k}}{\mathrm{d}x^{2k}}$ over the interval $(0,1)$ after discretizing them on a uniform grid of step size $h = \frac{1}{(n+1)}$. The difficulty of the problem addressed therein lies in the fact that not only the order of the matrices $T_n(a_n)$ depends on n but also their symbols a_n.

The main results are stated in Section 2. If we abbreviate $\lambda_j\big(T_n(a_n)\big)$ to $\lambda_{j,n}$, recall that $0 < \lambda_{1,n} < \lambda_{2,n} < \cdots < \lambda_{n,n}$ for all n, and notice that as j may also depend on n we actually study sequences of the form $\lambda_{j_n,n}$ as $n \to \infty$, we refer to $\lambda_{j_n,n}$ as:

- *extreme eigenvalues* if $\frac{j_n}{\sqrt{n}} \to 0$;
- *inner eigenvalues* if all we know is that $j_n \to \infty$;
- *strictly inner eigenvalues* if even $\frac{j_n}{\sqrt{n}} \to \infty$.

The main results – a triplet of theorems – are third order asymptotic formulas for the eigenvalues of $T_n(a_n)$. Theorem 2.1 considers the case of extreme eigenvalues, Theorem 2.2 addresses the case of inner eigenvalues, and Theorem 2.3 deals with the case of strictly inner eigenvalues.

The proof of this triplet of results occupy much of the paper. In Section 3, the equations behind the main results are derived. Technical aspects of the proofs are presented in Section 4 while the core of the proofs are given in Section 5. The sixth and last Section contains some selected numerical experiments of three types. The first type concerns inner eigenvalues, while the second type concerns extreme eigenvalues. As for the numerical experiment of the third type, they provide a comparison with a formula by Seymour V. Parter.

Reviewer: Frédéric Morneau-Guérin (Québec)

MSC:

15B05 Toeplitz, Cauchy, and related matrices
15A18 Eigenvalues, singular values, and eigenvectors
47B35 Toeplitz operators, Hankel operators, Wiener-Hopf operators

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