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# The norm of an infinite $L$-matrix

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## Acknowledgement

# Basic definitions

$L$-matrices

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### Definition

An $L$-matrix is an infinite matrix which is of the form

$$A := [a_n] = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_1 & a_1 & a_2 & a_3 & \cdots \\ a_2 & a_2 & a_2 & a_3 & \cdots \\ a_3 & a_3 & a_3 & a_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

## Basic definitions

*L*-matrices

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### Definition

An *L*-matrix is an infinite matrix which is of the form

$$A := [a_n] = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_1 & a_1 & a_2 & a_3 & \cdots \\ a_2 & a_2 & a_2 & a_3 & \cdots \\ a_3 & a_3 & a_3 & a_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

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When does the *L*-matrix $[a_n]$ act as a bounded operator on $\ell^2$?

## Some examples
Cesàro matrices

### Definition

The Cesàro matrices $C_p$ are the matrices

$$C_p := \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ \left(\frac{1}{2}\right)^p & \left(\frac{1}{2}\right)^p & 0 & 0 & \cdots \\ \left(\frac{1}{3}\right)^p & \left(\frac{1}{3}\right)^p & \left(\frac{1}{3}\right)^p & 0 & \cdots \\ \left(\frac{1}{4}\right)^p & \left(\frac{1}{4}\right)^p & \left(\frac{1}{4}\right)^p & \left(\frac{1}{4}\right)^p & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

$C_p$ acts as a bounded operator on $\ell^2$ if $p \geq 1$. Moreover, $\|C_1\|_{\ell^2 \to \ell^2} = 2$.

## Some examples
Hilbert matrix

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### Definition

The infinite Hilbert matrix is

$$\mathcal{H} := \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \cdots \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

$\mathcal{H}$ acts as a bounded operator on $\ell^2$ and $\|\mathcal{H}\|_{\ell^2 \to \ell^2} = \pi$.

## Motivation
History lesson

The first[1] instance of a $L$-matrix being used is in 1983 in *Tricks or treats with the Hilbert matrix*.

The author notices that $\mathcal{H} \leq [a_n] = C_1 C_1^*$ where $a_n = \frac{1}{n+1}$.

$\implies \|\mathcal{H}\|_{\ell^2 \to \ell^2} \leq \|[a_n]\|_{\ell^2 \to \ell^2} \leq \|C_1\|_{\ell^2 \to \ell^2} \|C_1^*\|_{\ell^2 \to \ell^2} = 4.$

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[1] To the best of our knowledge.

# Motivation
### Weighted Dirichlet spaces

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#### Definition

Let $\omega$ be a positive superharmonic function on $\mathbb{D}$ and $f \in \mathrm{Hol}(\mathbb{D})$. We define

$$\mathcal{D}_\omega(f) := \int_{\mathbb{D}} |f'(z)|^2 \omega(z) \mathrm{d}A(z),$$

where $\mathrm{d}A$ denotes normalized area measure on $\mathbb{D}$. The *weighted Dirichlet space* $\mathcal{D}_\omega$ is the set of functions $f \in \mathrm{Hol}(\mathbb{D})$ such that $\mathcal{D}_\omega(f) < \infty$.

## Motivation
Hadamard multipliers

### Definition

Let $f(z) := \sum_{k=0}^{\infty} a_k z^k$ and $g(z) := \sum_{k=0}^{\infty} b_k z^k$ be two formal power series. Their *Hadamard product* is defined to be

$$(f * g)(z) := \sum_{k=0}^{\infty} a_k b_k z^k.$$

The *Hadamard mulipliers* of $\mathcal{D}_\omega$ are the formal power series $h$ that have the property that $h * f \in \mathcal{D}_\omega$ for each $f \in \mathcal{D}_\omega$.

## Motivation
Definition of $T_h$

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### Definition

Let $h(z) := \sum_{k=0}^{\infty} c_k z^k$. We define

$$
T_h := \begin{pmatrix}
c_1 & c_2 - c_1 & c_3 - c_2 & c_4 - c_3 & \dots \\
0 & c_2 & c_3 - c_2 & c_4 - c_3 & \dots \\
0 & 0 & c_3 & c_4 - c_3 & \dots \\
0 & 0 & 0 & c_4 & \dots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
$$

## Motivation
Characterization of the Hadamard multipliers of $\mathcal{D}_\omega$

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### Theorem (Mashreghi–Ransford, 2019)

*Let $h(z)$ be a formal power series. The following statements are equivalent.*

*(i) $h$ is an Hadamard multiplier of $\mathcal{D}_\omega$ for every superharmonic weight $\omega$.*

*(ii) $T_h$ acts as a bounded operator on $\ell^2$.*

*Moreover, $\mathcal{D}_\omega(h * f) \leq \|T_h\|^2_{\ell^2 \to \ell^2} \mathcal{D}_\omega(f)$ for every superharmonic weight $\omega$ and $\|T_h\|^2_{\ell^2 \to \ell^2}$ is the best possible constant.*

## Theorem 1
Particular case

### Theorem 1

Let $A = [a_n]$ be an $L$-matrix such that $(a_n)_{n \geq 0}$ is a positive strictly decreasing sequence. Suppose that

$$\Delta := \sup_{n \in \mathbb{N}} \frac{a_n (a_n + a_{n-1})}{a_{n-1} - a_n} < \infty.$$

Then $A$ acts as a bounded operator on $\ell^2$ and $\|A\|_{\ell^2 \to \ell^2} \leq 2 \max(a_0, \Delta)$.

# Theorem 1
### General case

### Theorem 1

Let $A = [a_n]$ be a positive L-matrix and $(\delta_n)_{n \geq 0}$ be a positive strictly decreasing sequence. Suppose that

$$\Delta := \sup_{n \in \mathbb{N}} \frac{(a_n + \delta_{n-1})(a_n + \delta_n)}{\delta_{n-1} - \delta_n} < \infty.$$

Then $A$ acts as a bounded operator on $\ell^2$ and
$\|A\|_{\ell^2 \to \ell^2} \leq \max(\delta_0 + a_0, \Delta)$.

## Theorem 2
Definition

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**Definition**

We say that the sequence $(a_n)$ is *lacunary* if there is a constant $\rho > 1$ and a subsequence $(n_j)$ such that $n_{j+1}/n_j \geq \rho$ and $a_n = 0$ except possibly for indices $n \in \{n_j : j \geq 1\}$.

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**Example**

$(a_n) = \left(1, \frac{1}{2}, 0, \frac{1}{4}, 0, 0, 0, \frac{1}{8}, 0, 0, 0, 0, 0, 0, 0, \frac{1}{16}, \dots\right)$ is a lacunary sequence.

# Theorem 2
Particular case (p=2)

### Theorem 2

Let $A = [a_n]$ be an L-matrix with lacunary coefficient $(a_n)$ satisfying

$$\sum_{s=j}^{\infty} \sqrt{n_s} |a_{n_s}|^2 = O(1/\sqrt{n_j}), \qquad (as\ j \to \infty).$$

Then $A$ maps $\ell^2$ to itself as a bounded operator.

# Theorem 2
General case

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### Theorem 2

Let $p > 1$, with $q$ such that $1 = \frac{1}{p} + \frac{1}{q}$, and let $A = [a_n]$ be an L-matrix with lacunary coefficient $(a_n)$ satisfying

$$\sum_{s=j}^{\infty} |a_{n_s}|^p n_s^{(1-t)p/q} = O(n_j^{-tp/q}), \qquad (j \to \infty),$$

for some $t \in [0, 1)$. Then $A$ maps $\ell^p$ to itself as a bounded operator.

# Theorem 2

General case (quantitative estimation)

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### Theorem 2

*Fix $p > 1$ and $t \in (0,1)$ and suppose we are under the assumptions of the last theorem. Moreover, suppose that the lacunary sequence has a ratio $\rho > 1$. Then,*

$$\|A\|_{\ell^p \to \ell^p} \leq 2\eta \left( \frac{\rho^{1-t}}{\rho^{1-t} - 1} \right)^{1/q} + \|(a_n)\|_\infty,$$

*where* $\eta := \sup_{j \geq 1} \left( n_j^{tp/q} \sum_{s=j}^{\infty} |a_{n_s}|^p n_s^{(1-t)p/q} \right)^{1/p}$.

## Corollaries
Estimation of a nice lacunary sequence

### Corollary

*Let $N \geq 2$ be a positive integer and let $0 \leq R \leq 1/\sqrt{N}$. Let $A = [a_n]$ be the L-matrix with lacunary coefficient*

$$a_{N^j} = R^j, \qquad (j \geq 1),$$

*and $a_n = 0$ for other values of $n$. Then $A$ is a bounded operator on $\ell^2$.*

## Corollaries
Complete characterization of $O\left(n^{-\alpha}\right)$

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### Corollary

Let $A = [a_n]$ be a positive L-matrix. The condition
$a_n = O\left(1/n^{\alpha}\right)$ is

- necessary if $\alpha = \frac{1}{2}$;
- nor necessary, nor sufficient if $\frac{1}{2} < \alpha < 1$;
- sufficient if $\alpha = 1$;

for $A$ to act as a bounded operator on $\ell^2$.

# Complete characterization of $O\left(n^{-\alpha}\right)$

Demonstration of the "sufficient" part

Let $f(a_n, \delta_n) := \frac{(a_n + \delta_{n-1})(a_n + \delta_n)}{\delta_{n-1} - \delta_n}$.

$\Delta = \sup_{n \in \mathbb{N}} f(a_n, \delta_n) < \infty \iff \limsup_{n \to \infty} f(a_n, \delta_n) < \infty;$

## Complete characterization of $O\left(n^{-\alpha}\right)$

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- $\delta_n := \frac{K}{n};$

# Complete characterization of $O\left(n^{-\alpha}\right)$

### Demonstration of the "sufficient" part

Let $f(a_n, \delta_n) := \frac{(a_n + \delta_{n-1})(a_n + \delta_n)}{\delta_{n-1} - \delta_n}$.

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- $(a_n)_{n \geq 0} = O\left(\frac{1}{n}\right) \implies \exists K \in \mathbb{R}, a_n \leq \frac{K}{n} \quad (\forall n \geq 1);$

$\implies \limsup_{n \to \infty} f(a_n, \delta_n) \leq \limsup_{n \to \infty} f(\frac{K}{n}, \delta_n);$

- $\delta_n := \frac{K}{n};$

- $\limsup_{n \to \infty} f(\frac{K}{n}, \frac{K}{n}) = \lim_{n \to \infty} 2K\left(2 - \frac{1}{n}\right) = 4K < \infty.$

$\square$

## Corollaries
The norm of a particular set of $L$-matrices

> **Theorem**
>
> Let $A_s = [a_n]$ be the $L$-matrix defined by $a_n = \frac{1}{n+s}$, $(s \geq \frac{1}{2\sqrt{2}})$.
> Then $\|A_s\|_{\ell^2 \to \ell^2} = 4$.

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$\|A_s\|_{\ell^2 \to \ell^2} \geq \|A_s e_1\|_{\ell^2} \geq a_0 = 1/s.$

## Corollaries
The norm of a particular set of $L$-matrices

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$\|A_s\|_{\ell^2 \to \ell^2} \geq \|A_s e_1\|_{\ell^2} \geq a_0 = 1/s.$

What is the value of $s_0 := \inf\{s > 0 : \|A_s\|_{\ell^2 \to \ell^2} = 4\}$?

Basic definitions
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Motivation
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Theorem 1
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Theorem 2
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**Corollaries**
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## References

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