The Hilbert L-matrix and its generalizations

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Approximation of a continuous function

The problem

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The problem

Suppose that f is a continuous function on $[a, b]$.

Question : Which polynomial of degree $n - 1$ better approximate f on $[a, b]$ with respect to the $\mathcal{L}^2([a, b])$ norm?

$$\text{minimize} \qquad \int_a^b \Big(\textstyle\sum_{k=1}^n c_k x^{k-1} - f(x)\Big)^2 \mathrm{d}x \,.$$

Approximation of a continuous function

The solution

Hilbert (1894) showed that the coefficients of the polynomial are obtained by solving the system of linear equations

$$\mathcal{H}_n c \;=\; b\,,$$

where $c = (c_1, \ldots, c_n)^{tr}$, $b = (b_k)_{1 \leq k \leq n}$ is defined by $b_k = \int_a^b f(x) x^{k-1} \mathrm{d}x$ and $\mathcal{H}_n$ is the *Hilbert matrix* of order n.

The Hilbert matrix of order n

Definition

The *Hilbert matrix* of order n is the matrix

$$\mathcal{H}_n := \left[\frac{1}{i+j+1}\right]_{0 \le i,j \le n-1}.$$

Example

The Hilbert matrix of order 4 is

$$\mathcal{H}_4 = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{bmatrix}.$$

The Hilbert matrix

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The (infinite) *Hilbert matrix* is the matrix

$$\mathcal{H} := \left[\frac{1}{i+j+1}\right]_{i,j\geq 0} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \cdots \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

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$\mathcal{H}$ acts as a bounded operator on ℓ^2 and $\|\mathcal{H}\|_{\ell^2 \to \ell^2} = \pi$.

The Hilbert L-matrix

Definition

Let $s \in \mathbb{R} \setminus (-\mathbb{N}_0)$. The *Hilbert L-matrix* is the matrix

$$\mathcal{L}_s := \left[\frac{1}{\max\{i,j\} + s}\right]_{i,j \geq 0} = \begin{bmatrix} \frac{1}{s} & \frac{1}{1+s} & \frac{1}{2+s} & \frac{1}{3+s} & \cdots \\ \frac{1}{1+s} & \frac{1}{1+s} & \frac{1}{2+s} & \frac{1}{3+s} & \cdots \\ \frac{1}{2+s} & \frac{1}{2+s} & \frac{1}{2+s} & \frac{1}{3+s} & \cdots \\ \frac{1}{3+s} & \frac{1}{3+s} & \frac{1}{3+s} & \frac{1}{3+s} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The Cesàro matrix

Definition

The Cesàro matrix $\mathcal{C}$ is the matrix

$$\mathcal{C} := \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \cdots \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

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- $\mathcal{C}$ acts as a bounded operator on ℓ^2 and $\|\mathcal{C}\|_{\ell^2 \to \ell^2} = 2$.

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- $\mathcal{C}$ acts as a bounded operator on ℓ^2 and $\|\mathcal{C}\|_{\ell^2 \to \ell^2} = 2$.
- The Hilbert L-matrix $\mathcal{L}_1$ satisfies $\mathcal{L}_1 = \mathcal{C}\mathcal{C}^*$.

Motivation

(and some history)

The first[1] instance of an L-matrix being used is by Choi in 1983 in *Tricks or treats with the Hilbert matrix*.

The author notices that $\mathcal{H} \leq \mathcal{L}_1 = \mathcal{C}\mathcal{C}^*$.

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$$\begin{aligned} \implies \quad \|\mathcal{H}\|_{\ell^2\to\ell^2} &\leq \|\mathcal{L}_1\|_{\ell^2\to\ell^2} \\ &\leq \|\mathcal{C}\|_{\ell^2\to\ell^2}\|\mathcal{C}^*\|_{\ell^2\to\ell^2} \\ &= \|\mathcal{C}\|_{\ell^2\to\ell^2}\|\mathcal{C}\|_{\ell^2\to\ell^2} \\ &= 4. \end{aligned}$$

[1] To the best of our knowledge.

The L-matrices

Definition

Let (a_n) be a sequence of complex number. An L-matrix is an infinite matrix of the form

$$A := [a_n] = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_1 & a_1 & a_2 & a_3 & \cdots \\ a_2 & a_2 & a_2 & a_3 & \cdots \\ a_3 & a_3 & a_3 & a_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

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Question : When does the L-matrix $[a_n]$ act as a bounded operator on ℓ^2 and what is its norm?

Motivation

Weighted Dirichlet spaces

Definition

Let ω be a positive superharmonic function on $\mathbb{D}$ and $f \in \mathrm{Hol}(\mathbb{D})$. We define

$$\mathcal{D}_\omega(f) := \int_{\mathbb{D}} |f'(z)|^2 \omega(z) \mathrm{d}A(z),$$

where $\mathrm{d}A$ denotes normalized area measure on $\mathbb{D}$. The *weighted Dirichlet space* $\mathcal{D}_\omega$ is the set of functions $f \in \mathrm{Hol}(\mathbb{D})$ such that $\mathcal{D}_\omega(f) < \infty$.

Motivation

Hadamard multipliers

Definition

Let $f(z) := \sum_{k=0}^{\infty} a_k z^k$ and $g(z) := \sum_{k=0}^{\infty} b_k z^k$ be two formal power series. Their *Hadamard product* is defined to be

$$(f * g)(z) := \sum_{k=0}^{\infty} a_k b_k z^k.$$

The *Hadamard multipliers* of $\mathcal{D}_\omega$ are the formal power series h which have the property that $h * f \in \mathcal{D}_\omega$ for each $f \in \mathcal{D}_\omega$.

Motivation

Characterization of the Hadamard multipliers of $\mathcal{D}_\omega$

Theorem (Mashreghi–Ransford, 2019)

Let $h(z)$ be a formal power series. TFAE.

(i) h is an Hadamard multiplier of $\mathcal{D}_\omega$ for every superharmonic weight ω.

(ii) The associated matrix T_h acts as a bounded operator on ℓ^2.

(iii) The associated L-matrix L_h acts as a bounded operator on ℓ^2.

Moreover, $\mathcal{D}_\omega(h * f) \leq \|T_h\|^2_{\ell^2\to\ell^2}\mathcal{D}_\omega(f)$ for every superharmonic weight ω and $\|T_h\|^2_{\ell^2\to\ell^2}$ is the best possible constant.

A sufficient condition

Particular case

Theorem (B.–Mashreghi, 2021)

Let $A = [a_n]$ be an L-matrix such that $(|a_n|)_{n\geq 0}$ is a strictly decreasing sequence. Suppose that

$$\Delta := \sup_{n\in\mathbb{N}} \left(|a_n| \cdot \frac{|a_{n-1}| + |a_n|}{|a_{n-1}| - |a_n|} \right) < \infty.$$

Then A acts as a bounded operator on ℓ^2 and $\|A\|_{\ell^2\to\ell^2} \leq 2\max\left(|a_0|, \Delta\right)$.

A sufficient condition

General case

Theorem (B.–Mashreghi, 2021)

Let $A = [a_n]$ be an L-matrix and $(\delta_n)_{n\geq 0}$ be a positive strictly decreasing sequence. Suppose that

$$\Delta := \sup_{n\in\mathbb{N}} \frac{(|a_n| + \delta_{n-1})(|a_n| + \delta_n)}{\delta_{n-1} - \delta_n} < \infty.$$

Then A acts a bounded operator on ℓ^2 and $\|A\|_{\ell^2\to\ell^2} \leq \max(|a_0| + \delta_0, \Delta)$.

The norm of the Hilbert L-matrix
(Part 1)

- $\|\mathcal{L}_1\|_{\ell^2\to\ell^2} \leq \|\mathcal{C}\|_{\ell^2\to\ell^2}\|\mathcal{C}^*\|_{\ell^2\to\ell^2} = 4$

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Theorem (B.–Mashreghi, 2021)

Suppose that $s \geq \frac{\sqrt{2}}{4}$. *Then* $\|\mathcal{L}_s\|_{\ell^2\to\ell^2} = 4.$

An interesting phenomenon

- $\|\mathcal{L}_s\|_{\ell^2 \to \ell^2} \geq \|\mathcal{L}_s e_1\|_{\ell^2} \geq a_0 = \frac{1}{s}$.

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Question: What is the value of $s_0 := \inf\{s > 0 : \|\mathcal{L}_s\|_{\ell^2 \to \ell^2} = 4\}$?

An interesting phenomenon

• $\|\mathcal{L}_s\|_{\ell^2\to\ell^2} \geq \|\mathcal{L}_s e_1\|_{\ell^2} \geq a_0 = \frac{1}{s}$.

Question: What is the value of $s_0 := \inf\{s > 0 : \|\mathcal{L}_s\|_{\ell^2\to\ell^2} = 4\}$?

Theorem (B.–Mashreghi, 2022)

Let $s_0 = \inf\{s > 0 : \|\mathcal{L}_s\|_{\ell^2\to\ell^2} = 4\}$. *Then*

$$0.347 \approx \frac{\sqrt{48+18\sqrt{3}} - \sqrt{3} - 3}{12} \leqslant s_0 \leqslant \frac{\sqrt{2}}{4} \approx 0.353.$$

Important observation

Theorem (F. Štampach, 2021)

Let (a_n) be a sequence of real number such that $a_{n+1} \neq a_n$ for every n. Then the formal inverse of the L-matrix $A = [a_n]$ is the Jacobi matrix

$$B = \begin{bmatrix} b_0 & -b_0 & & & & \\ -b_0 & b_0 + b_1 & -b_1 & & & \\ & -b_1 & b_1 + b_2 & -b_2 & & \\ & & -b_2 & b_2 + b_3 & -b_3 & \\ & & & \ddots & \ddots & \ddots \end{bmatrix},$$

where $b_n = 1/(a_n - a_{n+1})$, $n \in \mathbb{N}_0$.

The value of s_0

Let ${}_3F_2\left(\begin{matrix} a,\ b,\ c \\ d,\ e \end{matrix} \,\middle|\, z\right)$ denote the hypergeometric ${}_3F_2$-functions.

Theorem (F. Štampach, 2021)

Let $s_0 = \inf\{s > 0 : \|\mathcal{L}_s\|_{\ell^2 \to \ell^2} = 4\}$*. Then* s_0 *is the unique positive zero of the function*

$$s \ \mapsto \ {}_3F_2\left(\begin{matrix} -1/2,\ 1/2,\ 3/2 \\ 1,\ s+1/2 \end{matrix} \,\middle|\, 1\right).$$

Numerically, we have $s_0 \approx 0.349086$.

The norm of the Hilbert L-matrix
(Part 2)

Theorem (F. Štampach, 2021)

If $0 < s < s_0$*, then* $\|\mathcal{L}_s\|_{\ell^2 \to \ell^2} = \frac{4}{1-4x^2(s)}$*, where* $x(s)$ *is the unique zero in* $(0, 1/2)$ *of the function*

$$t \ \mapsto \ {}_3F_2\left(\begin{matrix} t-1/2,\ t+1/2,\ t+3/2 \\ 2t+1,\ t+s+1/2 \end{matrix} \,\middle|\, 1 \right).$$

The norm of the Hilbert L-matrix

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Remark

The methods of Štampach also work for negative s *as long as* $s \notin -\mathbb{N}_0$. *In these cases, he conjectured that there is at most two distinct eigenvalue of* $\mathcal{L}_s$.

An extension to the ℓ^p space

Theorem (B.–Mashreghi, 2021)

If $s \geq 1$ and $1 < p < \infty$, then $\|\mathcal{L}_s\|_{\ell^p \to \ell^p} = \frac{p^2}{p-1}$.

An extension to the ℓ^p space

Theorem (B.–Mashreghi, 2021)

If $s \geq 1$ and $1 < p < \infty$, then $\|\mathcal{L}_s\|_{\ell^p \to \ell^p} = \frac{p^2}{p-1}$.

Question: What is the value of $s_0(p) := \inf \left\{ s > 0 : \|\mathcal{L}_s\|_{\ell^p \to \ell^p} = \frac{p^2}{p-1} \right\}$?

References

- Mashreghi, J., Ransford, T. (2019). Hadamard multipliers on weighted Dirichlet spaces. *Integral Equations and Operator Theory*, 91(6), 1-13.
- Bouthat, L., Mashreghi, J. (2021) The norm of an infinite L-matrix. *Operators and Matrices*. 15(4) : 47-58
- Bouthat, L., Mashreghi, J. (2021) L-Matrices with Lacunary Coefficients. *Operators and Matrices*. 15(65) : 1045-1053
- Bouthat, L., Mashreghi, J. (2022) The critical point and the p-norm of the Hilbert L-matrix. *Linear Algebra and Its Applications*. 634 : 1-14
- Štampach, F. (2022) The Hilbert L-matrix. arXiv preprint arXiv:2107.10694.