Some results about infinite L-matrices

Ludovick Bouthat

Université Laval

June 10, 2021

Acknowledgement

Part of this research was done under the NSERC research grant with the supervision of Pr. Javad Mashreghi.

Basic definitions

L-matrices

Definition

An L-matrix is an infinite matrix which is of the form

$$A := [a_n] = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_1 & a_1 & a_2 & a_3 & \cdots \\ a_2 & a_2 & a_2 & a_3 & \cdots \\ a_3 & a_3 & a_3 & a_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Basic definitions

L-matrices

Definition

An L-matrix is an infinite matrix which is of the form

$$A := [a_n] = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_1 & a_1 & a_2 & a_3 & \cdots \\ a_2 & a_2 & a_2 & a_3 & \cdots \\ a_3 & a_3 & a_3 & a_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

When does the L-matrix $[a_n]$ act as a bounded operator on ℓ^2?

Some examples

Cesàro matrices

Definition

The Cesàro matrices C_p are the matrices

$$C_p := \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ \left(\frac{1}{2}\right)^p & \left(\frac{1}{2}\right)^p & 0 & 0 & \cdots \\ \left(\frac{1}{3}\right)^p & \left(\frac{1}{3}\right)^p & \left(\frac{1}{3}\right)^p & 0 & \cdots \\ \left(\frac{1}{4}\right)^p & \left(\frac{1}{4}\right)^p & \left(\frac{1}{4}\right)^p & \left(\frac{1}{4}\right)^p & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

C_p acts as a bounded operator on ℓ^2 if $p \geq 1$. Moreover, $\|C_1\|_{\ell^2 \to \ell^2} = 2$.

Some examples

Hilbert matrix

Definition

The infinite Hilbert matrix is

$$\mathcal{H} := \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \cdots \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

$\mathcal{H}$ acts as a bounded operator on ℓ^2 and $\|\mathcal{H}\|_{\ell^2 \to \ell^2} = \pi$.

Motivation

History lesson

The first[1] instance of a L-matrix being used is in 1983 in *Tricks or treats with the Hilbert matrix.*

The author notices that $\mathcal{H} \leq [a_n] = C_1 C_1^*$ where $a_n = \frac{1}{n+1}$.

$$\implies \|\mathcal{H}\|_{\ell^2 \to \ell^2} \leq \|[a_n]\|_{\ell^2 \to \ell^2} \leq \|C_1\|_{\ell^2 \to \ell^2} \|C_1^*\|_{\ell^2 \to \ell^2} = 4.$$

[1] To the best of our knowledge.

Motivation

Weighted Dirichlet spaces

Definition

Let ω be a positive superharmonic function on $\mathbb{D}$ and $f \in \mathrm{Hol}(\mathbb{D})$. We define

$$\mathcal{D}_\omega(f) := \int_{\mathbb{D}} |f'(z)|^2 \omega(z) \mathrm{d}A(z),$$

where $\mathrm{d}A$ denotes normalized area measure on $\mathbb{D}$. The *weighted Dirichlet space* $\mathcal{D}_\omega$ is the set of functions $f \in \mathrm{Hol}(\mathbb{D})$ such that $\mathcal{D}_\omega(f) < \infty$.

Motivation

Hadamard multipliers

Definition

Let $f(z) := \sum_{k=0}^{\infty} a_k z^k$ and $g(z) := \sum_{k=0}^{\infty} b_k z^k$ be two formal power series. Their *Hadamard product* is defined to be

$$(f * g)(z) := \sum_{k=0}^{\infty} a_k b_k z^k.$$

The *Hadamard mulipliers* of $\mathcal{D}_\omega$ are the formal power series h that have the property that $h * f \in \mathcal{D}_\omega$ for each $f \in \mathcal{D}_\omega$.

Motivation

Definition of T_h

Definition

Let $h(z) := \sum_{k=0}^{\infty} c_k z^k$. We define

$$T_h := \begin{pmatrix} c_1 & c_2 - c_1 & c_3 - c_2 & c_4 - c_3 & \dots \\ 0 & c_2 & c_3 - c_2 & c_4 - c_3 & \dots \\ 0 & 0 & c_3 & c_4 - c_3 & \dots \\ 0 & 0 & 0 & c_4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Motivation

Characterization of the Hadamard multipliers of $\mathcal{D}_\omega$

Theorem (Mashreghi–Ransford, 2019)

Let $h(z)$ be a formal power series. The following statements are equivalent.
(i) h is an Hadamard multiplier of $\mathcal{D}_\omega$ for every superharmonic weight ω.
(ii) T_h acts as a bounded operator on ℓ^2.

Moreover, $\mathcal{D}_\omega(h * f) \leq \|T_h\|^2_{\ell^2\to\ell^2}\mathcal{D}_\omega(f)$ for every superharmonic weight ω and $\|T_h\|^2_{\ell^2\to\ell^2}$ is the best possible constant.

Theorem 1

Particular case

Theorem 1

Let $A = [a_n]$ be an L-matrix such that $(|a_n|)_{n \geq 0}$ is a strictly decreasing sequence. Suppose that

$$\Delta := \sup_{n \in \mathbb{N}} \left(|a_n| \cdot \frac{|a_{n-1}| + |a_n|}{|a_{n-1}| - |a_n|} \right) < \infty.$$

Then A acts as a bounded operator on ℓ^2 and
$\|A\|_{\ell^2 \to \ell^2} \leq 2 \max\left(|a_0|, \Delta\right).$

Theorem 1

General case

Theorem 1

Let $A = [a_n]$ *be an L-matrix and* $(\delta_n)_{n\geq 0}$ *be a positive strictly decreasing sequence. Suppose that*

$$\Delta := \sup_{n\in\mathbb{N}} \frac{(|a_n| + \delta_{n-1})(|a_n| + \delta_n)}{\delta_{n-1} - \delta_n} < \infty.$$

Then A *acts as a bounded operator on* ℓ^2 *and* $\|A\|_{\ell^2\to\ell^2} \leq \max(|a_0| + \delta_0, \Delta)$.

Theorem 2

Definition

Definition

We say that the sequence (a_n) is *lacunary* if there is a constant $\rho > 1$ and a subsequence (n_j) such that $n_{j+1}/n_j \geq \rho$ and $a_n = 0$ except possibly for indices $n \in \{n_j : j \geq 1\}$.

Example

$(a_n) = \left(1, \frac{1}{2}, 0, \frac{1}{4}, 0, 0, 0, \frac{1}{8}, 0, 0, 0, 0, 0, 0, 0, \frac{1}{16}, \dots\right)$ is a lacunary sequence.

Theorem 2

Particular case (p=2)

Theorem 2

Let $A = [a_n]$ be an L-matrix with lacunary coefficient (a_n) satisfying

$$\sum_{s=j}^{\infty} \sqrt{n_s}|a_{n_s}|^2 = O(1/\sqrt{n_j}), \qquad (as\ j \to \infty).$$

Then A maps ℓ^2 to itself as a bounded operator.

Theorem 2

General case

Theorem 2

Let $p > 1$, with q such that $1 = \frac{1}{p} + \frac{1}{q}$, and let $A = [a_n]$ be an L-matrix with lacunary coefficient (a_n) satisfying

$$\sum_{s=j}^{\infty} |a_{n_s}|^p n_s^{(1-t)p/q} = O(n_j^{-tp/q}), \qquad (j \to \infty),$$

for some $t \in [0, 1)$. Then A maps ℓ^p to itself as a bounded operator.

Theorem 2

General case (quantitative estimation)

Theorem 2

Fix $p > 1$ and $t \in (0,1)$ and suppose we are under the assumptions of the last theorem. Moreover, suppose that the lacunary sequence has a ratio $\rho > 1$. Then,

$$\|A\|_{\ell^p \to \ell^p} \leq 2\eta \left(\frac{\rho^{1-t}}{\rho^{1-t} - 1} \right)^{1/q} + \|(a_n)\|_\infty,$$

where $\eta := \sup_{j \geq 1} \left(n_j^{tp/q} \sum_{s=j}^{\infty} |a_{n_s}|^p n_s^{(1-t)p/q} \right)^{1/p}$.

Corollaries

Estimation of a nice lacunary sequence

Corollary

Let $N \geq 2$ *be a positive integer and let* $0 \leq R \leq 1/\sqrt{N}$. *Let* $A = [a_n]$ *be the L-matrix with lacunary coefficient*

$$a_{N^j} = R^j, \qquad (j \geq 1),$$

and $a_n = 0$ *for other values of* n. *Then* A *is a bounded operator on* ℓ^2.

Corollaries

Estimation of a nice lacunary sequence

Corollary

Let $N \geq 2$ be a positive integer and let

$$B := \begin{pmatrix} a_0 & 0 & 0 & 0 & \cdots \\ a_1 & a_1 & 0 & 0 & \cdots \\ a_2 & a_2 & a_2 & 0 & \cdots \\ a_3 & a_3 & a_3 & a_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $a_{N^j} = N^{\frac{j}{2}}$ for $j \geq 1$ and $a_n = 0$ for other values of n. Then B is a bounded operator on ℓ^2 and $\|B\|_{\ell^2 \to \ell^2} = \frac{\sqrt{N-1}}{\sqrt{N}-1}$.

Corollaries

Complete characterization of $O\left(n^{-\alpha}\right)$

Corollary

Let $A = [a_n]$ be a positive L-matrix. The condition $a_n = O\left(1/n^{\alpha}\right)$ is

- *necessary if $\alpha = \frac{1}{2}$;*
- *nor necessary, nor sufficient if $\frac{1}{2} < \alpha < 1$;*
- *sufficient if $\alpha = 1$;*

for A to act as a bounded operator on ℓ^2.

Corollaries

The norm of a particular set of L-matrices

Theorem

Let $A_s = [a_n]$ *be the L-matrix defined by* $a_n = \frac{1}{n+s}$, $(s \geq \frac{\sqrt{2}}{4})$. *Then* $\|A_s\|_{\ell^2 \to \ell^2} = 4$.

Corollaries

The norm of a particular set of L-matrices

Theorem

Let $A_s = [a_n]$ *be the* L*-matrix defined by* $a_n = \frac{1}{n+s}$, $(s \geq \frac{\sqrt{2}}{4})$.
Then $\|A_s\|_{\ell^2 \to \ell^2} = 4$.

Theorem

Let $A_s = [a_n]$ *be the* L*-matrix defined by* $a_n = \frac{1}{n+s}$, $(s \geq \mathbf{1})$.
Then $\|A_s\|_{\ell^p \to \ell^p} = \frac{p^2}{p-1}$.

Corollaries

Open question and partial results

$$\|A_s\|_{\ell^2\to\ell^2} \geq \|A_s e_1\|_{\ell^2} \geq a_0 = 1/s.$$

Corollaries

Open question and partial results

$$\|A_s\|_{\ell^2 \to \ell^2} \geq \|A_s e_1\|_{\ell^2} \geq a_0 = 1/s.$$

What is the value of $s_0 := \inf\{s > 0 : \|A_s\|_{\ell^2 \to \ell^2} = 4\}$?

Corollaries

Open question and partial results

$$\|A_s\|_{\ell^2\to\ell^2} \geq \|A_s e_1\|_{\ell^2} \geq a_0 = 1/s.$$

What is the value of $s_0 := \inf\{s > 0 : \|A_s\|_{\ell^2\to\ell^2} = 4\}$?

Theorem

Let $s_0 = \inf\{s > 0 : \|A_s\|_{\ell^2\to\ell^2} = 4\}$. *Then*

$$0.347 \approx \frac{\sqrt{48 + 18\sqrt{3}} - \sqrt{3} - 3}{12} \leqslant s_0 \leqslant \frac{\sqrt{2}}{4} \approx 0.353.$$

References

- Ludovick Bouthat, Javad Mashreghi. *On the norm of L-matrices*, 2021.
- Ludovick Bouthat, Javad Mashreghi. *L–matrices with lacunary coefficients*, Accepted.
- Javad Mashreghi, Thomas Ransford. *Hadamard multipliers on weighted Dirichlet spaces*, 2019.