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A study of the validity of Oppenheim's inequality for Hurwitz matrices associated with Hurwitz polynomials. (English. English summary)

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A *Hurwitz polynomial* is a polynomial whose roots lie in the open left half of the complex plane. Such a polynomial must have coefficients that are positive real numbers.

Let $\mathcal{P}_n$ denote the family of all polynomials of degree n with positive coefficients, and let $\mathcal{H}_n$ be the family of all Hurwitz polynomials in $\mathcal{P}_n$.

Given a polynomial

$$p(x)=\sum_{k=0}^{n} a_k x^k,$$

where the a_is are positive numbers, the *Hurwitz matrix* $H(p)$ associated with p is given by

$$H(p)=\begin{bmatrix} a_{n-1} & a_{n-3} & a_{n-5} & \dots & 0 \\ a_n & a_{n-2} & a_{n-4} & \dots & 0 \\ 0 & a_{n-1} & a_{n-3} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_0 \end{bmatrix}.$$

The leading principal minors of the matrix $H(p)$ are given by the following determinants, called *Hurwitz determinants*,

$$\begin{aligned} \Delta_1 &:= a_{n-1}, \\ \Delta_2 &:= \begin{vmatrix} a_{n-1} & a_{n-3} \\ a_n & a_{n-2} \end{vmatrix}, \\ \Delta_3 &:= \begin{vmatrix} a_{n-1} & a_{n-3} & a_{n-5} \\ a_n & a_{n-2} & a_{n-4} \\ 0 & a_{n-1} & a_{n-3} \end{vmatrix}, \\ &\vdots \\ \Delta_n &:= \det(H(p)). \end{aligned}$$

Recall that for two matrices $A=[a_{ij}]$ and $B=[b_{ij}]$ of the same dimension $m\times n$, the Hadamard product $A\circ B$ is a matrix of the same dimension as the operands, with elements given by $A\circ B=[a_{ij}b_{ij}]$. Likewise, for two polynomials

$$p(x)=\sum_{k=0}^{n} a_k x^k \text{ and } q(x)=\sum_{k=0}^{n} b_k x^k$$

of equal degree n, then the *Hadamard product* $p\circ q$ is the polynomial of the same dimension as the operands and defined by

$$(p\circ q)(x):=\sum_{k=0}^{n} a_k b_k x^k.$$

A notable inequality (sometimes referred to as Oppenheim's inequality) regarding the Hadamard product of two positive semidefinite matrices $A=[a_{ij}]$ and $B=[b_{ij}]$ of order n states that

$$\det(A\circ B)\geq \det(A)\cdot\det(B).$$

In the paper under review, the authors study the problem of whether this inequality holds for Hurwitz matrices. Their main result stipulates that it does if $n = 3, 4, 5$ (albeit with a strict inequality). They also obtain a generalization (requiring the addition of a term to the left-hand member for which an explicit formula is given) for the case $n = 6$.

This main result is largely based on other results obtained previously, including one in which the Hurwitz matrix for $p \in \mathcal{P}_n$ is associated with a positive definite matrix of roughly the half order and one in which relations for the Hurwitz determinants of a Hurwitz matrix for $p \in \mathcal{P}_n$ are obtained. In the process, the authors obtain a new necessary and sufficient condition for a polynomial to be Hurwitz.

The body of the text presents both the main results and some explanatory examples.

At the very end of the paper, the authors present an application of their main results: they derive a necessary and sufficient condition for the Hadamard square root of a Hurwitz polynomial of degree 5 to be Hurwitz. *Frédéric Morneau-Guérin*

[References]

1. B.A. Asner, Jr. On the total nonnegativity of the Hurwitz matrix. *SIAM J. Appl. Math.*, 18(2):407–414, 1970. MR0260768
2. S.P. Bhattacharyya, H. Chapellat, and L.H. Keel. *Robust Control: The Parametric Approach.* Prentice Hall, Upper Saddle River, NJ, 1995.
3. S. Białas, L. Białas-Cież, and M. Kudra. On the Hurwitz stability of noninteger Hadamard powers of stable polynomials. *Linear Algebra Appl.*, 683:111–124, 2014. MR4677715
4. A.S. Crans, S.M. Fallat, and C.R. Johnson. The Hadamard core of the totally nonnegative matrices. *Linear Algebra Appl.*, 328(1–3):203–222, 2001. MR1823517
5. K.B. Dwelle. *Some Results on Hadamard Closure and Variation Diminishing Properties of Totally Nonnegative Matrices.* PhD thesis, Purdue University, 2007. MR2711691
6. S.M. Fallat and C.R. Johnson. *Totally Nonnegative Matrices.* Princeton Series in Applied Mathematics. Princeton University Press, Princeton and Oxford, 2011. MR2791531
7. J. Garloff and D.G. Wagner. Hadamard products of stable polynomials are stable. *J. Math. Anal. Appl.*, 202(3):797–809, 1996. MR1408355
8. M. Gasca, C.A. Micchelli, and J.M. Peña. Almost strictly totally positive matrices. *Numer. Algorithms*, 2:225–236, 1992. MR1165907
9. J.H.B. Kemperman. A Hurwitz matrix is totally positive. *SIAM J. Math. Anal.*, 13(2):331–341, 1982. MR0647131
10. Y.-T. Li and J.-C. Li. On the estimations of bounds for determinant of Hadamard product of H-matrices. *J. Comput. Math.*, 19(4):365–370, 2001. MR1842849
11. A Liénard and M.H. Chipart. Sur le signe de la partie réelle des racines d'une équation algébraique. *J. Math. Pures Appl.*, 10(6):291–346, 1914.
12. M. Lin. An Oppenheim type inequality for a block Hadamard product. *Linear Algebra Appl.*, 452:1–6, 2014. MR3201085
13. J. Liu and L. Zhu. Some improvement of Oppenheim's inequality for M-matrices. *SIAM J. Matrix Anal. Appl.*, 18(2):305–311, 1997. MR1437332
14. A. Oppenheim. Inequalities connected with definite Hermitian forms. *J. Lond. Math. Soc.*, s1-5(2):114–119, 1930. MR1574213
15. F. Zhang. *Matrix Theory: Basic Results and Techniques.* 2nd ed., Springer, New York, NY, 2011. MR2857760
16. X.-D. Zhang and C.-X. Ding. The equality cases for the inequalities of Oppenheim and Schur for positive semi-definite matrices. *Czechoslowak Math. J.*, 59(134):197–206, 2009. MR2486625

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.