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Estimates of the p-norms of solutions and inverse matrices of systems of linear equations with a circulant matrix. (English. English summary)

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Recall that an $n \times n$ circulant matrix A takes the form

$$A = \operatorname{circ}(a_0, a_1, \ldots, a_{n-1}) = \begin{pmatrix} a_0 & a_1 & \ldots & a_{n-2} & a_{n-1} \\ a_{n-1} & a_0 & \ldots & a_{n-3} & a_{n-2} \\ \ldots & a_{n-1} & a_0 & \ddots & \ldots \\ a_2 & \ddots & \ddots & \ddots & a_1 \\ a_1 & a_2 & \ldots & a_{n-1} & a_0 \end{pmatrix}.$$

The polynomial $Q(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1}$ is called the *associated polynomial* of the matrix A.

This article considers the problem of estimating solutions and inverse matrices of systems of linear equations with a circulant matrix in the matrix norms induced by the vector p-norm for $1 < p < \infty$, i.e.,

$$\|B\|_p := \sup_{y \neq 0} \frac{\|By\|_p}{\|y\|_p}.$$

Calculating or obtaining estimates of such norms for $p \notin \{1, 2, \infty\}$ is not an easy task. But certain exact values and estimates are known for certain special matrices (as the authors are quick to point out, citing all the relevant references, including some fairly recent ones).

The main result of Section 2 is as follows:

If a circulant matrix A is diagonally dominant, i.e., if, for some $k \in \{0, 1, \ldots, n\}$,

$$|a_k| - \sum_{\substack{j=0 \\ j \neq k}}^{n} |a_j| = r > 0,$$

then the system of linear equations $Ax = b$ has a unique solution x. Moreover, the following estimate holds:

$$\|x\|_p \leq \frac{\|b\|_p}{r}.$$

From the straightforward proof of this theorem one can immediately derive an estimate for the p-norm of the inverse of the matrix A:

$$\|A^{-1}\|_p \leq \frac{1}{r}.$$

Section 3 focuses on arbitrary nondegenerate circulant matrices. Borrowing the idea of decomposing a circulant matrix into a product of matrices associated with factors in the decomposition of the associated polynomial $Q(z)$ (an idea developed and exploited by the first of the two authors in a previous publication), estimates for the solution of the system of linear equations $Ax = b$ in the matrix norms induced by the vector p-norms $1 < p < \infty$ are obtained as well as estimates for the inverse matrix A^{-1}.

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