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A certain Bruhat order on doubly substochastic matrices. (English)

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The symmetric groupe of degree n, denoted by S_n is the set of all permutations of the symbols $1, 2, \dots n$. It is well-known that the symmetric group S_n can be identified bijectively with $\mathcal{P}_n$, the set of all $n \times n$ permutation matrices, using the association

$$\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n) \longleftrightarrow P_\sigma = [p_{i,j}],$$

where

$$p_{i,j} = \begin{cases} 1, & \text{if } j = \sigma_i \\ 0, & \text{otherwise.} \end{cases} \quad (1 \le i, j \le n)$$

An *inversion* of a permutation $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$ is a pair (σ_u, σ_v) such that $u < v$ but $\sigma_u > \sigma_v$.

The symmetric group S_n can be endowed with a partial order called the *Bruhat order* and denoted $\preceq_B$. Given two permutations σ and τ, we say that $\sigma \preceq_B \tau$ if σ can be obtained from τ by a sequence of operations which interchanges the two entries of an inversion. The Bruhat order on permutations can also be reformulated in a number of ways, i.e., we say that $\sigma \preceq_B \tau$ if and only if any of the following equivalent statements holds true:

- For all $1 \le k \le n$, $\sigma[k] \preceq_G \tau[k]$, where $\preceq_G$ denotes the *Gale order* and $\sigma[k], \tau[k]$ are the lists of the first k elements in σ and τ respectively.
- The permutation matrix P_σ can be obtained from P_τ by sequentially replacing a 2×2 submatrix $L_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ with $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
- The matrix $\Sigma(P_\sigma) - \Sigma(P_\tau)$ is nonnegative, where $\Sigma(A)$ denotes the matrix whose (i, j)-th entry is $\sum_{1 \le k \le i,\ 1 \le l \le j} a_{k,l}$.

In the paper under consideration, we look at a generalization of permutations and permutation matrices called *subpermutation* and *subpermutation matrix*. A subpermutation is an n-tuple $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$ such that $\sigma_i \in \{1, 2, \dots, n\} \cup \{\star\}$ for all $1 \le i \le n$, where each number in $\{1, 2, \dots, n\}$ appears at most once in σ. A subpermutation matrix is a $(0, 1)$-matrix such that each row and column contains at most one 1.

In Section 2, the author defines a generalized Bruhat order on subpermutations in such a way that this definition coincides with the Bruhat order on permutations. This is achieved in two ways: firstly by employing a generalization of the Gale order and secondly by using partial sum matrices. The section ends with a long, explicit example.

In Section 3, it is shown that this generalized Bruhat order is isomorphic to the inverse of the adherence order. It is thus graded and lexicographically shellable.

In Section 4, following in the footsteps of Brualdi and Dahl, the author studies the Bruhat shadow of a subpermutation matrix. This section mainly contains results that are used in the following section.

In Section 5, the generalized Bruhat order is further extended to provide a partial order – called the *substochastic Bruhat order* – on the set ω_n of $n \times n$ doubly substochastic matrices (i.e., nonnegative matrices each of whose rows and columns sums at most to 1). The notion of Bruhat faces on ω_n is then introduced. Subsequently, relations between Bruhat faces on the Birkhoff polytope Ω_n (consisting of all the $n \times n$ doubly stochastic matrices) and those on ω_n are presented. These relations are thenused to determine whether a given subpermutation matrix induces a Bruhat face.

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MSC:

15B51	Stochastic matrices
05B20	Combinatorial aspects of matrices (incidence, Hadamard, etc.)
06A07	Combinatorics of partially ordered sets
52B05	Combinatorial properties of polytopes and polyhedra (number of faces, shortest paths, etc.)

Keywords:
Bruhat order; Bruhat shadow; Bruhat faces; subpermutation matrices; doubly substochastic matrices

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