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Sub-defect of product of $I \times I$ finite sub-defect matrices. (English) Zbl 07895551

Linear Multilinear Algebra 72, No. 10, 1581-1589 (2024).

A square matrix of nonnegative real numbers is said to be *doubly stochastic* if the sum of the elements in each row and column is equal to 1, and *doubly* ***sub****stochastic* if the sum of the elements in each row and column is less than or equal to 1.

L. Cao et al. [Linear Multilinear Algebra 64, No. 11, 2313–2334 (2016; Zbl 1358.15025)] introduced the concept of the *sub-defect* of an $n \times n$ doubly substochastic matrix A, denoted by $\mathrm{sd}(A)$, which is defined as the minimum number of rows and columns needed to be added to A to obtain a doubly stochastic matrix D containing A as a submatrix, Moreover, they showed that the sub-defect of A can be easily calculated using the sum of all elements of A. Later on, *L. Cao* and *S. Koyuncu* [Linear Multilinear Algebra 65, No. 4, 653–657 (2017; Zbl 1367.15052)] went on to derive the following lower and upper bounds for the sub-defect of the product of two $n \times n$ doubly substochastic matrices A and B:

$$\max\{\mathrm{sd}(A), \mathrm{sd}(B)\} \;\le\; \mathrm{sd}(AB) \;\le\; \min\{n, \mathrm{sd}(A) + \mathrm{sd}(B)\}.$$

In the present article, the author turns his attention on the case of doubly substochastic matrices with infinitely (possibly uncountably so) many rows and columns. In this context, the sum of the elements along the rows or columns is to be interpreted as the supremum of the sums over a finite subset of the indexed family of numbers.

In order to briefly describe the main thrust of the article, a few definitions and explanations are necessary.

- an infinite matrix $[a_{ij}]_{i,j\in\mathcal{I}}$ is called a submatrix of $[b_{ij}]_{i,j\in\mathcal{I}\cup\mathcal{J}}$ if $a_{ij} = b_{ij}$ for all $i, j \in \mathcal{I}$;
- an $\mathcal{I} \times \mathcal{I}$ doubly stochastic matrix D is called a *completion* of an $\mathcal{I} \times \mathcal{I}$ doubly substochastic matrix A if A is a submatrix of D;
- the sub-defect of an infinite doubly substochastic matrix is defined as the minimum cardinal number α such that A has a $(\mathcal{I} \cup \mathcal{J}) \times (\mathcal{I} \cup \mathcal{J})$ completion D, where $\mathcal{J}$ is a set of cardinality α which is disjoint from $\mathcal{I}$;
- an infinite matrix A is *of finite sub-defect* if $\mathrm{sd}(A) < \aleph_0$.

The main result achieved by the author in the present paper is a direct and immediate analogue of that obtained by Cao & Koyuncu for product of doubly substochastic matrices. Indeed, he showed that if A, B are two $\mathcal{I} \times \mathcal{I}$ doubly substochastic matrices of finite sub-defect, then AB is also of finite sub-defect and

$$\max\{\mathrm{sd}(A), \mathrm{sd}(B)\} \;\le\; \mathrm{sd}(AB) \;\le\; \min\{\mathrm{card}(\mathcal{I}), \mathrm{sd}(A) + \mathrm{sd}(B)\}.$$

A final remark presents specific examples showing that both bounds are sharp.

Reviewer: Frédéric Morneau-Guérin (Québec)

MSC:

15B51	Stochastic matrices
15A83	Matrix completion problems
47B80	Random linear operators
03E10	Ordinal and cardinal numbers

Keywords:
completion; finite sub-defect; increasable; doubly substochastic; cardinal number

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