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Inversion formulas for Toeplitz-plus-Hankel matrices. (English) Zbl 07874841

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The main motivation for the questions considered in this paper comes from results by Gohberg and Semencul (also sometimes written Sementsul) in 1972. The pair derived an inversion formula for Toeplitz matrices

$$T_n(\mathbf{a}) = [a_{i-j}]_{i,j=0}^{n-1}, \qquad \mathbf{a} = (a_j)_{j=-n+1}^{n-1}.$$

This formula expresses $T_n(\mathbf{a})^{-1}$ as a sum of products of triangular Toeplitz matrices involving only the entries of the first and last column (or row) of $T_n(\mathbf{a})^{-1}$. Such inversion formulas can also be established for Hankel matrices

$$H_n(\mathbf{b}) = [b_{i+j-n+1}]_{i,j=0}^{n-1}, \qquad \mathbf{b} = (b_j)_{j=-n+1}^{n-1}.$$

The central goal of the present paper is to establish inversion formulas of Gohberg–Semencul type for matrices which are the sum of a Toeplitz and a Hankel matrix (such matrices are called *Toeplitz-plus-Hankel matrices* or $T+H$ matrices), i.e.,

$$TH_n(\mathbf{a},\mathbf{b}) = T_n(\mathbf{a}) + H_n(\mathbf{b}).$$

The main results are presented in Section 3 and Section 4. In the first of these two sections, they authors obtain invertibility criteria for the $n \times n$ $T+H$ matrix $TH_n(\mathbf{a},\mathbf{b})$ and establish recursion formulas for all columns (or rows) of $TH_n(\mathbf{a},\mathbf{b})^{-1}$. These results are not new, but the direct proof is original. The second of these two sections begin with a reminder of a key 1988 discovery of Heinig and the second author of the paper under consideration that the inverse of an invertible $T+H$ matrix is an invertible $T+H$-Bezoutian, and *vice versa*. Recall that a matrix $B = [b_{ij}]_{i,j=0}^{n-1}$ is called a *Toeplitz-plus-Hankel-Bezoutian* (henceforth $T+H$-Bezoutian) if there exist eight vectors $\mathbf{u}_i, \mathbf{v}_i \in \mathbb{F}^{n+2} (i = 1,2,3,4)$ such that, in polynomial language,

$$B(t,s) = \sum_{i,j=0}^{n-1} b_{ij} t^i s^j = \frac{\sum_{i=1}^{4} \mathbf{u}_i(t)\mathbf{v}_i(s)}{(t-s)(1-ts)}.$$

Following this, the authors derive invertibility criteria and establish inversion formulas for $TH_n(\mathbf{a},\mathbf{b})$, where only the entries of four columns (or/and four rows) of $TH_n(\mathbf{a},\mathbf{b})^{-1}$ are involved. This is however shown to come at the expense that a certain 2×2 matrix has to be nonsingular. Lastly, the authors derive inversion formulas of Gohberg–Semencul type for the $(n-2)\times(n-2)$ $T+H$ matrix obtained from $TH_n(\mathbf{a},\mathbf{b})$ by deleting the first and the last column and the first and the last row

In Section 5 we look at a simple example of size $n=5$ illustrating the recursion formula for the columns of $TH_n(\mathbf{a},\mathbf{b})^{-1}$ as well as some results stated in Section 4.

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MSC:

15A09 Theory of matrix inversion and generalized inverses
15B05 Toeplitz, Cauchy, and related matrices
65F05 Direct numerical methods for linear systems and matrix inversion

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Toeplitz matrix; Hankel matrix; Toeplitz-plus-Hankel matrix; matrix inversion

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