Monotonicité de certaines sommes de Riemann

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Outline of the presentation

1. Theoretical motivation for the problem;
2. Explanation of the problem;
3. Some history and past results;
4. Recent development;
5. Some application to the initial motivation.

Doubly stochastic matrices

Definition

A square matrix is *doubly stochastic* if:

- nonnegative coefficients;
- row sums $= 1$;
- column sums $= 1$.

The set of doubly stochastic matrices of order n is denoted by Ω_n.

The diameter of Ω_n

The *diameter* of Ω_n relative to the Schatten p-norms $(1 \leq p \leq 2)$ satisfy

$$\operatorname{diam}_{\mathcal{S}_p}(\Omega_n) \;\geq\; 2\left(\sum_{k=1}^{n} \sin^p\left(\frac{k\pi}{n}\right)\right)^{1/p}.$$

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To prove that this bound is in fact an equality, we needed to show that

$$\frac{1}{n} \sum_{k=1}^{n} \sin^p\left(\tfrac{k\pi}{n}\right)$$

is a monotonically increasing function relative to n.

The right Riemann sum of f

Definition

Let $f : [0,1] \to \mathbb{R}$ be a Riemann integrable function. Then

$$R_n(f) := \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)$$

is the *right Riemann sum of f* over $[0,1]$ and $R_n(f) \xrightarrow{n\to\infty} \int_0^1 f(x)\mathrm{d}x$.

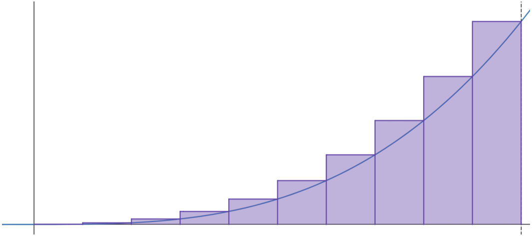

Figure: The right Riemann sum of $f(x) = x^3$ over $[0,1]$ when $n = 10$.

The left Riemann sum of f

Definition

Let $f : [0,1] \to \mathbb{R}$ be a Riemann integrable function. Then

$$L_n(f) := \frac{1}{n} \sum_{k=0}^{n-1} f\left(\frac{k}{n}\right)$$

is the *left Riemann sum of f* over $[0,1]$ and $L_n(f) \xrightarrow{n\to\infty} \int_0^1 f(x)\mathrm{d}x$.

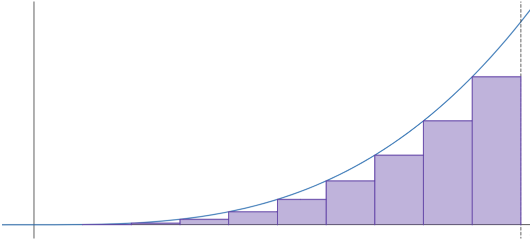

Figure: The left Riemann sum of $f(x) = x^3$ over $[0,1]$ when $n = 10$.

Some examples

The function $f(x) = x$

- If $f(x) = x$, then $L_n(f) = \frac{1}{2}\left(1 - \frac{1}{n}\right)$ and $R_n(f) = \frac{1}{2}\left(1 + \frac{1}{n}\right)$.

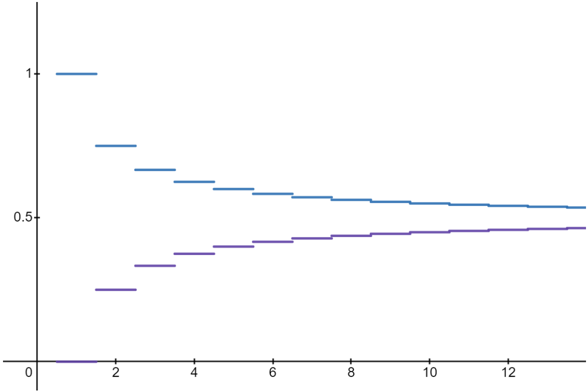

Figure: The left and right Riemann sum of $f(x) = x$ over $[0, 1]$.

Some examples

The function $f(x) = 5x - 4x^2 - \frac{2}{3}$

- If $f(x) = 5x - 4x^2 - \frac{2}{3}$, then $L_n(f) = \frac{3n^2-3n-4}{6n^2}$ and $R_n(f) = \frac{3n^2+3n-4}{6n^2}$.

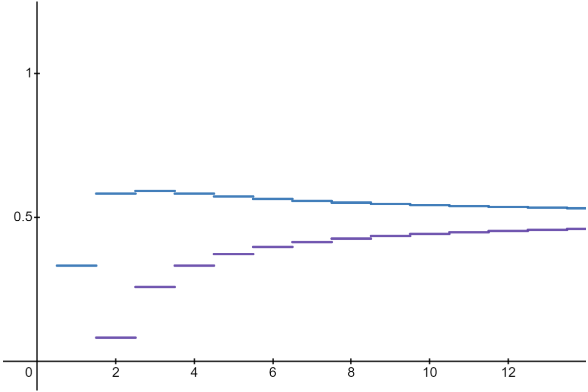

Figure: The left and right Riemann sum of $f(x) = 5x - 4x^2 - \frac{2}{3}$ over $[0, 1]$.

The function $f(x) = \frac{1}{1+x^2}$

- In 2012, Szilárd András asked if $L_n\left(\frac{1}{1+x^2}\right) = \sum_{k=0}^{n-1} \frac{n}{n^2+k^2}$ and $R_n\left(\frac{1}{1+x^2}\right) = \sum_{k=1}^{n} \frac{n}{n^2+k^2}$ exhibit some monotonicity properties.

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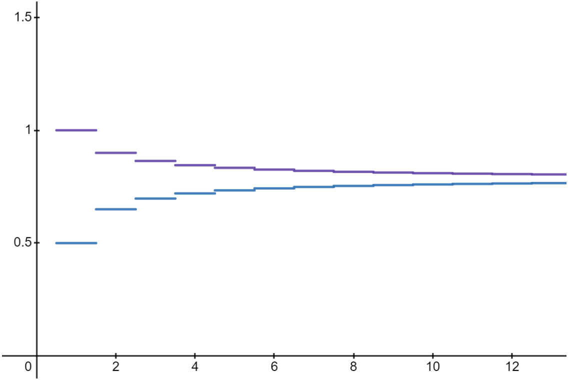

Figure: The left and right Riemann sum of $f(x) = \frac{1}{1+x^2}$ over $[0, 1]$.

Some general result using convexity

Theorem (S. András; 2012)

If $f : [0,1] \to \mathbb{R}$ is convex (or concave) and decreasing on the interval $[0,1]$, then $L_n(f)$ decreases monotonically and $R_n(f)$ increases monotonically relative to n.

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- Using the fact that $L_n(-f) = -L_n(f)$ and $R_n(-f) = -R_n(f)$, we also have:

Corollary (S. András; 2012)

If $f : [0,1] \to \mathbb{R}$ is convex (or concave) and increasing on the interval $[0,1]$, then $L_n(f)$ increases monotonically and $R_n(f)$ decreases monotonically relative to n.

A minor blunder

- András then asserts that the previous theorem apply to $f(x) = \frac{1}{1+x^2}$ and thus that $L_n(f)$ decreases monotonically and $R_n(f)$ increases monotonically relative to n.

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- András then asserts that the previous theorem apply to $f(x) = \frac{1}{1+x^2}$ and thus that $L_n(f)$ decreases monotonically and $R_n(f)$ increases monotonically relative to n. *However, f has an inflection point at $1/\sqrt{3}$.*

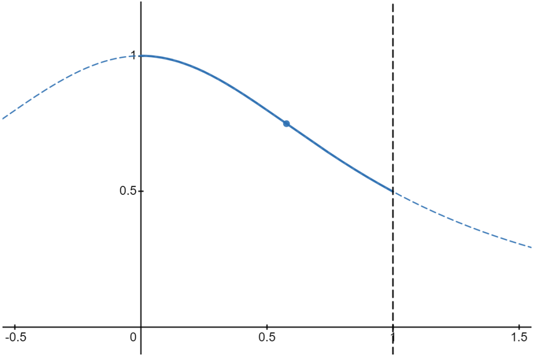

Figure: The function $f(x) = \frac{1}{1+x^2}$ and its inflection point at $1/\sqrt{3}$.

A (partial) solution

- This problem caught the attention of David Borwein and his son. They provided a rectified proof of the fact that $R_n\left(\frac{1}{1+x^2}\right) = \sum_{k=1}^{n} \frac{n}{n^2+k^2}$ increases monotonically.

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- To achieve this, they prove a series of theorems and corollaries which can be viewed as extensions of the theorems of S. András.

Some extensions

Theorem (D. Borwein *et al.*; 2020)

If the function $f : [0,1] \to \mathbb{R}$ *is convex on the interval* $[0,c]$ *for some* $0 < c < 1$, *concave on* $[c,1]$, *and decreasing on* $[0,1]$, *then* $R_n(f)$ *increases monotonically and* $L_n(f)$ *decreases monotonically relative to* n.

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Remark

One might expect this result to hold even when exchanging the roles of *convex* and *concave*. However, it suffice to consider $f(x) = 1_{[0,1/2]}$ to see that it cannot work, since

$$R_{2n-1}(f) + \frac{1}{2(n-1)} \;=\; R_{2n}(f) \;=\; R_{2n+1}(f) + \frac{1}{2n}.$$

Some extensions

- Considering $-f$ in the previous theorem yield:

Corollary (D. Borwein *et al.*; 2020)

If the function $f : [0,1] \to \mathbb{R}$ is concave on the interval $[0,c]$ for some $0 < c < 1$, convex on $[c,1]$, and increasing on $[0,1]$, then $R_n(f)$ decreases monotonically and $L_n(f)$ increases monotonically relative to n.

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- Using similar techniques, Borwein *et al.* showed that:

Theorem (D. Borwein *et al.*; 2020)

If the function $f : [0,1] \to \mathbb{R}$ is concave on the interval $[0,1]$, with maximum $f(c)$ for some $0 < c < 1$, then $R_n(f) - \frac{f(c)-f(0)}{n}$ increases monotonically relative to n.

Symmetrization (about $x = 1/2$)

Definition

Given a function $f : [0,1] \to \mathbb{R}$, its *symmetrization (about $x = \frac{1}{2}$)* is defined to be

$$\mathcal{F}_{1/2}(x) := \mathcal{F}(x) = \frac{f(x) + f(1-x)}{2}.$$

• The symmetrization of a convex (resp. concave) function is again convex (resp. concave). Interestingly, though the symmetrization process cannot destroy convexity or concavity, it can generate either of these properties.

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- The symmetrization of a convex (resp. concave) function is again convex (resp. concave). Interestingly, though the symmetrization process cannot destroy convexity or concavity, it can generate either of these properties.
- Such is the case of the function $f(x) = \frac{1}{1+x^2}$.

Some results using Symmetrization

Theorem (D. Borwein *et al.*; 2020)

If $f : [0,1] \to \mathbb{R}$ has a concave symmetrization and verifies $f(0) > f(\frac{1}{2})$, then $R_n(f)$ increases monotonically relative to n.

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If $f : [0,1] \to \mathbb{R}$ has a concave symmetrization and verifies $f(0) > f(\frac{1}{2})$, then $R_n(f)$ increases monotonically relative to n.

• Observing that $R_n(f(1-x)) = L_n(f(x))$, we obtain, by applying this theorem to $-f(x)$, $f(1-x)$, and $-f(1-x)$ respectively, the following corollaries:

Some results using Symmetrization

Corollary (D. Borwein *et al.*; 2020)

If f has a convex symmetrization and verifies $f(0) < f(\frac{1}{2})$, then $R_n(f)$ decreases monotonically relative to n.

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If f has a convex symmetrization and verifies $f(\frac{1}{2}) > f(1)$, then $L_n(f)$ decreases monotonically relative to n.

Application to $f(x) = \frac{1}{1+x^2}$

- The symmetrization of $f(x) = \frac{1}{1+x^2}$ is concave and satisfy $f(0) > f(1/2)$. Hence, $R_n(f)$ increases monotonically relative to n.

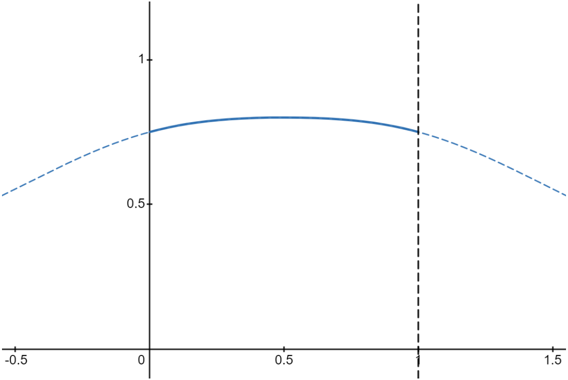

Figure: The symmetrization $\mathcal{F}(x)$ of the function $f(x) = \frac{1}{1+x^2}$.

The problem of $L_n(f)$

• Surprisingly, none of the above theorems allows us to prove directly that $L_n\left(\frac{1}{1+x^2}\right)$ decreases monotonically relative to n. Borwein *et al.* left it at that.

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• Recently, using the previous results as tools, we have been able to resolve the problem of $L_n(f)$ and even more, since we studied functions of the form

$$f_b(x) = \frac{1}{x^2 - bx + 1}, \qquad (b \leq 1).$$

The result and proof idea

Theorem

Let $f_b(x) = \frac{1}{x^2 - bx + 1}$ *with* $b \in \mathbb{R}$. *Then* $R_n(f_b)$ *increases monotonically relative to* n *for* $b \in (-\infty, 1]$ *and* $L_n(f_b)$ *decreases monotonically relative to* n *for* $b \in (-\infty, \alpha)$, *where* $\alpha \approx 0.493862$.

The result and proof idea

Theorem

Let $f_b(x) = \frac{1}{x^2 - bx + 1}$ with $b \in \mathbb{R}$. Then $R_n(f_b)$ increases monotonically relative to n for $b \in (-\infty, 1]$ and $L_n(f_b)$ decreases monotonically relative to n for $b \in (-\infty, \alpha)$, where $\alpha \approx 0.493862$.

Proof idea.

Let $g : [0, 1] \to \mathbb{R}$ and define the function $h := f_b - g$. Then $L_n(f_b) = L_n(h) + L_n(g)$ and $R_n(f_b) = R_n(h) + R_n(g)$.

Hence, we simply need to show that $L_n(h), L_n(g)$ decreases monotonically relative to n and that $R_n(h), R_n(g)$ increases monotonically relative to n.

Sketch of proof

Sketch of proof of $L_n(f_b)$.

For $b \in (-\infty, -1]$ the function $f_b(x)$ is convex and decreasing on $[0, 1]$ and one of the earlier theorem implies that $L_n(f_b)$ is monotonically decreasing.

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Otherwise, let

$$g_{a,c}(x) := \begin{cases} 0, & \text{if } 0 \leq x < \frac{1}{2}, \\ a\left(x - \frac{1}{2}\right)^2 + c\left(x - \frac{1}{2}\right)^4, & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

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For $b \in \left(-1, \frac{3-\sqrt{13}}{4}\right)$, let $a := 1 - b^2$ and $c := 0$.

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For $b \in \left[\frac{3-\sqrt{13}}{4}, \alpha\right)$ where $\alpha \approx 0.493862$, let $a := -32\frac{4b^2-6b-1}{(5-2b)^3}$ and $c := 16\left(\frac{59-198b+164b^2-40b^3}{(2-b)(5-2b)^3} - \varepsilon\right)$, where $\varepsilon > 0$.

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For $b \in \left[\frac{3-\sqrt{13}}{4}, 1\right]$, let $a := 32\frac{1+6b-4b^2}{(5-2b)^3}$, $c := -\frac{128}{3}\frac{1+6b-4b^2}{(5-2b)}$ and $d := \frac{512}{15}\frac{1+6b-4b^2}{(5-2b)^3}$.

The function $\sin^p(\pi x)$

- Recall our initial motivation: showing that

$$R_n\big(\sin^p(\pi x)\big) \;=\; \frac{1}{n}\sum_{k=1}^{n} \sin^p\big(\tfrac{k\pi}{n}\big)$$

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- Surprisingly, none of the above methods and theorems were able to provide a proof of this property. Nonetheless, we were able to show that $R_n\big(\sin^p(\pi x)\big)$ is indeed a monotonically increasing function relative to n for $p \in [1, 2]$.

"Sketch" of proof

- Using a myriad of identities, we were able to reduce the expression $R_{n+1}\big(\sin^p(\pi x)\big) - R_n\big(\sin^p(\pi x)\big)$ to a convenient form. These identities includes

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$$\sum_{k=0}^{n-1} \cos^{2j}\left(\tfrac{k\pi}{n}\right) = \frac{n}{4^j} \sum_{k=-\lfloor j/n \rfloor}^{\lfloor j/n \rfloor} \binom{2j}{j+kn}, \qquad (n, j \in \mathbb{N});$$

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$$(1+x)^z = \sum_{k=0}^{\infty} \binom{z}{k} x^k, \qquad (\mathrm{Re}(z) > 0, |x| \leq 1);$$

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$$(1+x)^z = \sum_{k=0}^{\infty} \binom{z}{k} x^k, \qquad (\mathrm{Re}(z) > 0, |x| \leq 1);$$

$$\sum_{j=0}^{\infty} \binom{p/2}{j} \binom{-1/2}{-1/2-j} = \binom{p/2-1/2}{-1/2}, \qquad (p \geq 0).$$

"Sketch" of proof

- We showed that

$$R_{n+1}\left(\sin^p(\pi x)\right) - R_n\left(\sin^p(\pi x)\right) \geq \sum_{j=n+1}^{\infty} B_j C_j$$

where

$$B_j := \frac{2}{4^j}\frac{\Gamma(j-p/2)}{j!\Gamma(-p/2)},$$

$$C_j := \sum_{k=1}^{\lfloor j/(n+1)\rfloor}\left(\binom{2j}{j+k(n+1)} - \binom{2j}{j+kn}\right).$$

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For $j \geq n+1$ and $p \in [0,2]$, we proved that $B_j, C_j \leq 0$. Hence, $R_{n+1}\big(\sin^p(\pi x)\big) \geq R_n\big(\sin^p(\pi x)\big)$.

Final application

Theorem

The diameter of Ω_n, the set of doubly stochastic matrices of order n, relative to the Schatten p-norms $(1 \leq p \leq 2)$ satisfy

$$diam_{\mathcal{S}_p}(\Omega_n) \;=\; 2\left(\sum_{k=1}^{n} \sin^p\left(\frac{k\pi}{n}\right)\right)^{1/p}.$$

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András, Szilárd (2012) Monotonicity of certain Riemann-type sums. *The Teaching Of Mathematics.*, 15(2): 113–120.

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