Monotonicity of Certain Riemann Sums

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Outline of the presentation

1. Theoretical motivation for the problem;
2. Explanation of the problem;
3. Some history and past results;
4. Recent development;
5. Some application to the initial motivation.

Doubly stochastic matrices

Definition

A square matrix is *doubly stochastic* if:

- nonnegative coefficients;
- row sums $= 1$;
- column sums $= 1$.

The set of doubly stochastic matrices of order n is denoted by Ω_n.

Example

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}.$$

The diameter of Ω_n

The *diameter of* Ω_n relative to the Schatten p-norms $(1 \leq p \leq 2)$ satisfy

$$\mathrm{diam}_{\mathcal{S}_p}(\Omega_n) \;\geq\; 2\left(\sum_{k=1}^{n} \sin^p\left(\frac{k\pi}{n}\right)\right)^{1/p}.$$

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To prove that this is an equality, we need to show that

$$\frac{1}{n}\sum_{k=1}^{n} \sin^p\left(\tfrac{k\pi}{n}\right)$$

is a **monotonically increasing function** relative to n.

The right Riemann sum of f

Definition

Let $f : [0,1] \to \mathbb{R}$ be a Riemann integrable function. Then

$$R_n(f) := \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)$$

is the *right Riemann sum of f* over $[0,1]$ and $R_n(f) \xrightarrow{n\to\infty} \int_0^1 f(x)\mathrm{d}x$.

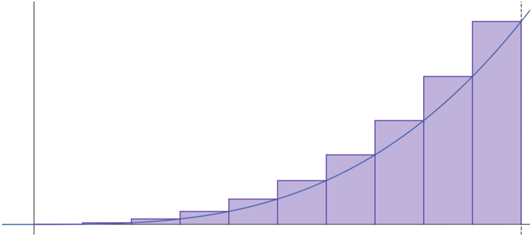

Figure: The right Riemann sum.

The left Riemann sum of f

Definition

Let $f : [0,1] \to \mathbb{R}$ be a Riemann integrable function. Then

$$L_n(f) := \frac{1}{n}\sum_{k=0}^{n-1} f\left(\frac{k}{n}\right)$$

is the *left Riemann sum of f* over $[0,1]$ and $L_n(f) \xrightarrow{n\to\infty} \int_0^1 f(x)\mathrm{d}x$.

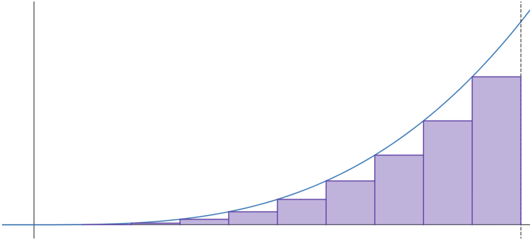

Figure: The left Riemann sum.

Some examples

The function $f(x) = x$

- If $f(x) = x$, then $L_n(f) = \frac{1}{2}\left(1 - \frac{1}{n}\right)$ and $R_n(f) = \frac{1}{2}\left(1 + \frac{1}{n}\right)$.

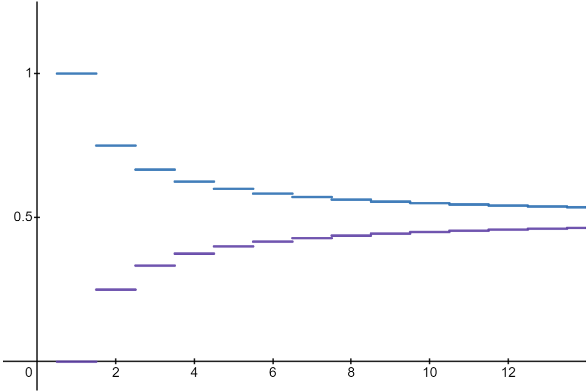

Figure: The left and right Riemann sum of $f(x) = x$ over $[0, 1]$.

Some examples

The function $f(x) = 5x - 4x^2 - \frac{2}{3}$

- If $f(x) = 5x - 4x^2 - \frac{2}{3}$, then $L_n(f) = \frac{3n^2-3n-4}{6n^2}$ and $R_n(f) = \frac{3n^2+3n-4}{6n^2}$.

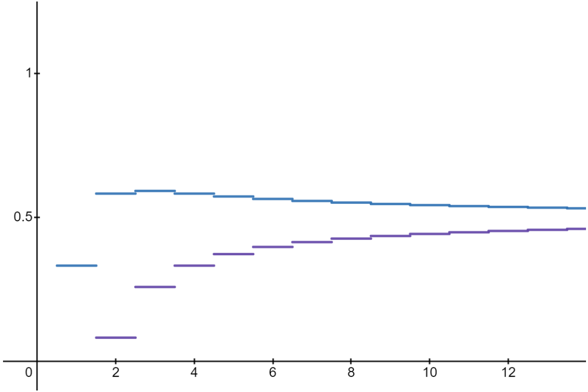

Figure: The left and right Riemann sum of $f(x) = 5x - 4x^2 - \frac{2}{3}$ over $[0, 1]$.

The function $f(x) = \frac{1}{1+x^2}$

- Szilárd András (2012) asked if $L_n\left(\frac{1}{1+x^2}\right) = \sum_{k=0}^{n-1} \frac{n}{n^2+k^2}$ and $R_n\left(\frac{1}{1+x^2}\right) = \sum_{k=1}^{n} \frac{n}{n^2+k^2}$ exhibit some monotonicity properties.

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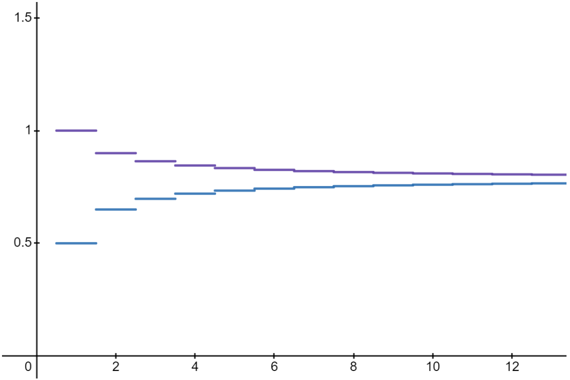

Figure: The left and right Riemann sum of $f(x) = \frac{1}{1+x^2}$ over $[0, 1]$.

Some general result using convexity

Theorem (S. András; 2012)

If $f : [0, 1] \to \mathbb{R}$ is convex (or concave) and decreasing on the interval $[0, 1]$, then $L_n(f)$ decreases monotonically and $R_n(f)$ increases monotonically relative to n.

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- Using the fact that $L_n(-f) = -L_n(f)$ and $R_n(-f) = -R_n(f)$, we also have:

Corollary (S. András; 2012)

If $f : [0,1] \to \mathbb{R}$ is convex (or concave) and increasing on the interval $[0,1]$, then $L_n(f)$ increases monotonically and $R_n(f)$ decreases monotonically relative to n.

A minor blunder

- András applied the last theorem to $f(x) = \frac{1}{1+x^2}$ and deduced that $L_n(f)$ decreases monotonically and $R_n(f)$ increases monotonically relative to n.

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• András applied the last theorem to $f(x) = \frac{1}{1+x^2}$ and deduced that $L_n(f)$ decreases monotonically and $R_n(f)$ increases monotonically relative to n. *However, f has an inflection point at $1/\sqrt{3}$.*

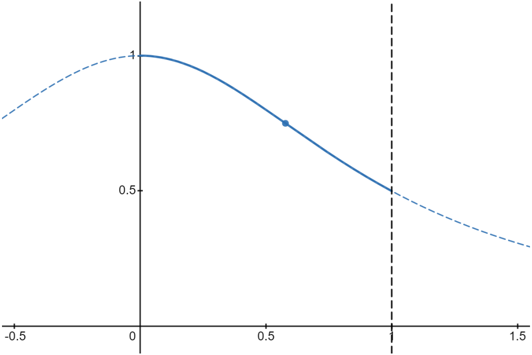

Figure: The function $f(x) = \frac{1}{1+x^2}$ and its inflection point at $1/\sqrt{3}$.

A (partial) solution

- This problem caught the attention of David Borwein and his son. They provided a rectified proof of the fact that $R_n\left(\frac{1}{1+x^2}\right) = \sum_{k=1}^{n} \frac{n}{n^2+k^2}$ increases monotonically.

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• To achieve this, they prove a series of theorems and corollaries which can be viewed as extensions of the theorems of S. András.

Some extensions

Theorem (Borwein, Borwein, Sims; 2020)

If the function $f : [0, 1] \to \mathbb{R}$ *is convex on the interval* $[0, c]$ *for some* $0 < c < 1$, *concave on* $[c, 1]$, *and decreasing on* $[0, 1]$, *then* $R_n(f)$ *increases monotonically and* $L_n(f)$ *decreases monotonically relative to* n.

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Theorem (Borwein, Borwein, Sims; 2020)

If the function $f : [0,1] \to \mathbb{R}$ *is convex on the interval* $[0,c]$ *for some* $0 < c < 1$*, concave on* $[c,1]$*, and decreasing on* $[0,1]$*, then* $R_n(f)$ *increases monotonically and* $L_n(f)$ *decreases monotonically relative to* n*.*

Remark

One might expect this result to hold even when exchanging the roles of *convex* and *concave*. However, it suffice to consider $f(x) = 1_{[0,1/2]}$ to see that it cannot work, since

$$R_{2n-1}(f) + \frac{1}{2(n-1)} \;=\; R_{2n}(f) \;=\; R_{2n+1}(f) + \frac{1}{2n}.$$

Some extensions

- Considering $-f$ in the previous theorem yield:

Corollary (Borwein, Borwein, Sims; 2020)

If the function $f : [0,1] \to \mathbb{R}$ is concave on the interval $[0,c]$ for some $0 < c < 1$, convex on $[c,1]$, and increasing on $[0,1]$, then $R_n(f)$ decreases monotonically and $L_n(f)$ increases monotonically relative to n.

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- Using similar techniques, the authors showed that:

Theorem (Borwein, Borwein, Sims; 2020)

If the function $f : [0,1] \to \mathbb{R}$ *is concave on the interval* $[0,1]$, *with maximum* $f(c)$ *for some* $0 < c < 1$, *then* $R_n(f) - \frac{f(c)-f(0)}{n}$ *increases monotonically relative to* n.

Symmetrization (about $x = 1/2$)

Definition

Given a function $f : [0,1] \to \mathbb{R}$, its *symmetrization (about $x = \frac{1}{2}$)* is defined to be

$$\mathcal{F}_{1/2}(x) := \mathcal{F}(x) = \frac{f(x) + f(1-x)}{2}.$$

- The symmetrization of a convex (resp. concave) function is again convex (resp. concave). Interestingly, though the symmetrization process cannot destroy convexity or concavity, it can generate either of these properties.

Some results using Symmetrization

Theorem (Borwein, Borwein, Sims; 2020)

If $f : [0,1] \to \mathbb{R}$ has a concave symmetrization and verifies $f(0) > f(\frac{1}{2})$, then $R_n(f)$ increases monotonically relative to n.

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- Observing that $R_n(f(1-x)) = L_n(f(x))$, we obtain, by applying this theorem to $-f(x)$, $f(1-x)$, and $-f(1-x)$ respectively, the following corollaries:

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Application to $f(x) = \frac{1}{1+x^2}$

- The symmetrization of $f(x) = \frac{1}{1+x^2}$ is concave and satisfy $f(0) > f(1/2)$. Hence, $R_n(f)$ increases monotonically relative to n.

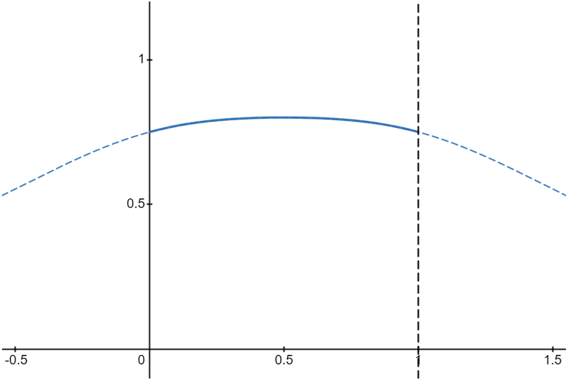

Figure: The symmetrization $\mathcal{F}(x)$ of the function $f(x) = \frac{1}{1+x^2}$.

The problem of $L_n(f)$

• Surprisingly, none of the above theorems allow us to prove that $L_n\left(\frac{1}{1+x^2}\right)$ decreases monotonically relative to n. Borwein *et al.* left it at that.

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• Recently, using the previous results as tools, we have been able to resolve the problem of $L_n(f)$ and even more, since we studied functions of the form

$$f_b(x) = \frac{1}{1 - bx + x^2}, \qquad (b \leq 1).$$

The result and proof idea

Theorem (B., Mashreghi, Morneau-Guérin; 2023)

If $f : [0,1] \to \mathbb{R}$ has a concave symmetrization and verifies $\max\{f(0), f(1)\} > f(\frac{1}{2})$, *then $R_n(f)$ increases monotonically relative to n.*

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Theorem (B., Mashreghi, Morneau-Guérin; 2023)

Let $f_b(x) = \frac{1}{x^2 - bx + 1}$ with $b \in \mathbb{R}$. Then $R_n(f_b)$ increases monotonically relative to n for $b \in (-\infty, 1]$ and $L_n(f_b)$ decreases monotonically relative to n for $b \in (-\infty, \frac{1}{2}]$. Moreover, these results are optimal.

The function $\sin^p(\pi x)$

- Recall our initial motivation: showing that

$$R_n\big(\sin^p(\pi x)\big) \;=\; \frac{1}{n}\sum_{k=1}^{n}\sin^p\left(\tfrac{k\pi}{n}\right)$$

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- Surprisingly, none of the above methods and theorems were able to provide a proof of this property. Nonetheless, we were able to show that $R_n\big(\sin^p(\pi x)\big)$ is indeed a monotonically increasing function relative to n for $p \in [1, 2]$.

Sketch of proof

- Using a myriad of identities, we were able to reduce the expression $R_{n+1}\big(\sin^p(\pi x)\big) - R_n\big(\sin^p(\pi x)\big)$ to a convenient form. These identities includes:

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$$\sum_{k=0}^{n-1} \cos^{2j}\left(\frac{k\pi}{n}\right) = \frac{n}{4^j} \sum_{k=-\lfloor j/n \rfloor}^{\lfloor j/n \rfloor} \binom{2j}{j+kn}, \qquad (n, j \in \mathbb{N});$$

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$$(1+x)^z = \sum_{k=0}^{\infty} \binom{z}{k} x^k, \qquad (\mathrm{Re}(z) > 0, |x| \leq 1);$$

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$$\sum_{j=0}^{\infty} \binom{p/2}{j} \binom{-1/2}{-1/2-j} = \binom{p/2-1/2}{-1/2}, \qquad (p \geq 0).$$

Sketch of proof

- We showed that

$$R_{n+1}\big(\sin^p(\pi x)\big) - R_n\big(\sin^p(\pi x)\big) \ \geq \ \sum_{j=n+1}^{\infty} B_j C_j$$

where

$$B_j \ := \ \frac{2}{4^j}\frac{\Gamma(j-p/2)}{j!\Gamma(-p/2)},$$

$$C_j \ := \ \sum_{k=1}^{\lfloor j/(n+1) \rfloor} \left(\binom{2j}{j+k(n+1)} - \binom{2j}{j+kn}\right).$$

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For $j \geq n+1$ and $p \in [0,2]$, each $B_j, C_j \leq 0$. Hence, $R_{n+1}\big(\sin^p(\pi x)\big) \geq R_n\big(\sin^p(\pi x)\big)$.

Final application

Theorem (B., Mashreghi, Morneau-Guérin; 2023)

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Corollary (B., Mashreghi, Morneau-Guérin; 2023)

The diameter of Ω_n, the set of doubly stochastic matrices of order n, relative to the Schatten p-norms $(1 \leq p \leq 2)$ satisfy

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