Les normes matricielles aléatoires: Une introduction à certains résultats modernes

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Stephan Ramon Garcia

Angel Chavez

My cat and I

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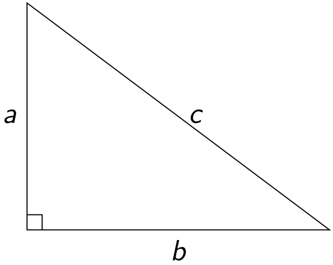
Bourses d'études supérieures du Canada
Vanier
Canada Graduate Scholarships

A triangle



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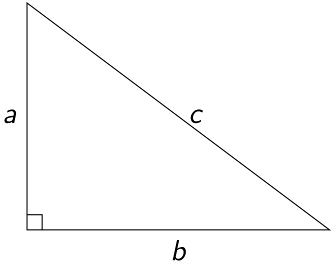

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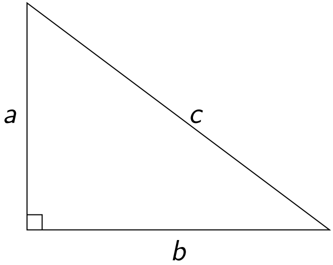

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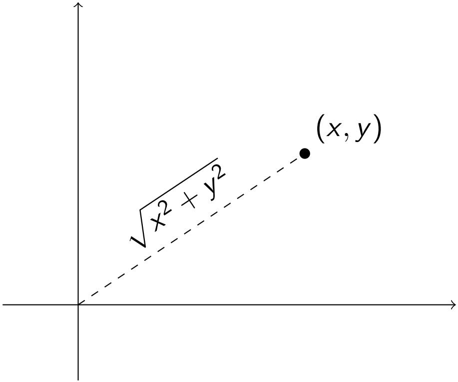
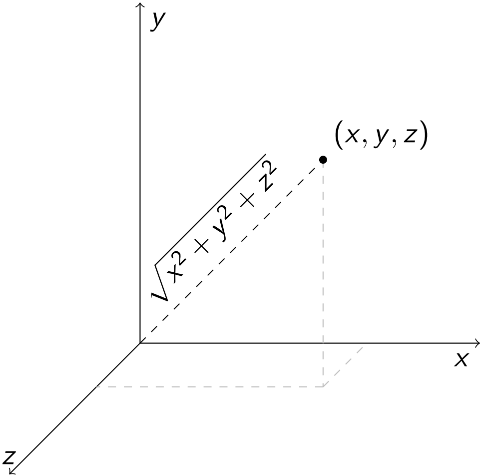
Pythagorean theorem

$$c = \sqrt{a^2 + b^2}$$

A notion of distance



What about 3 dimensions?



What about n dimensions ?

Definition

The *Euclidean distance* of a point $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ from the origin is defined by

$$\|x\|_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

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- This defines a *norm* on $\mathbb{R}^n$.

What is a norm ?

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If $x \in \mathbb{R}^n$, a norm is any function $f : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfying

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Example

If $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, then

$$\|x\|_p := \left(|x_1|^p + |x_2|^p + \cdots + |x_n|^p\right)^{\frac{1}{p}}$$

is a norm for any $p \in [1, \infty]$.

Generating matrix norms

• $f(\alpha x) = |\alpha| f(x)$ • $f(x+y) \leq f(x) + f(y)$ • $f(x) = 0 \iff x = 0$

- Consider $x := (x_1, x_2, \dots, x_{n^2}) \in \mathbb{R}^{n^2}$.

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- Rearrange x to get

$$X = \begin{bmatrix} x_1 & x_{n+1} & \cdots & x_{(n-1)n+1} \\ x_2 & x_{n+2} & \cdots & x_{(n-1)n+2} \\ \vdots & \vdots & \ddots & \vdots \\ x_n & x_{2n} & \cdots & x_{n^2} \end{bmatrix} \in \mathbb{R}^{n \times n}.$$

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$\implies \|X\| := \|x\|$ is a matrix norm.

An important property

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A function f defined on $n \times n$ real matrices is *submultiplicative* if for any $X, Y \in \mathbb{R}^{n \times n}$,

$$f(XY) \leq f(X) f(Y).$$

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• If an $n \times n$ matrix X is invertible and λ is an eigenvalue of X, then

$$\|X^{-1}\|^{-1} \leq |\lambda| \leq \|X\|.$$

An interesting example

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- Taking the p-norms $\|\cdot\|_p$ in $\mathbb{R}^{n^2}$ yield the *element-wise p-norms*

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Example

If $X = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$, then $\|X\|_p = 4^{1/p}$ and $\|X \cdot X\|_p = 2 \cdot 4^{1/p}$. Hence,

$$2 \cdot 4^{1/p} = \|X \cdot X\|_p \leq \|X\|_p \|X\|_p = 4^{2/p}$$

if and only if $p \leq 2$.

Observation

• $f(\alpha x) = |\alpha| f(x)$ • $f(x+y) \leq f(x) + f(y)$ • $f(x) = 0 \iff x = 0$ • $f(xy) \leq f(x)f(y)$

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- **Conclusion :** Identifying the matrix norms which are submultiplicative is an important problem.

A new surprising matrix norm

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Theorem (Chávez, Garcia, Hurley ; 2023)

Suppose that

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Then

$$\|A\|_{\boldsymbol{X},d} := \mathbb{E}\Big[|\langle \boldsymbol{X}, \boldsymbol{\lambda}\rangle|^d\Big]^{\frac{1}{d}} = \mathbb{E}\Big[|\lambda_1 X_1 + \lambda_2 X_2 + \cdots + \lambda_n X_n|^d\Big]^{\frac{1}{d}}$$

are matrix norms.

Some properties

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- **Question :** Under which conditions the norms $\|A\|_{\boldsymbol{X},d}$ are submultiplicative ?

Example ($d = 2$)

• $f(\alpha x) = |\alpha| f(x)$ • $f(x+y) \leq f(x) + f(y)$ • $f(x) = 0 \iff x = 0$ • $f(xy) \leq f(x)f(y)$
• $\|A\|_{\boldsymbol{X},d}^d = \mathbb{E}\big[|\lambda_1 X_1 + \lambda_2 X_2 + \cdots + \lambda_n X_n|^d\big]$

• If $d = 2$, μ is the mean and σ is the standard deviation of the distribution, then

$$\|A\|_{\boldsymbol{X},2} = \sqrt{\sigma^2 \|A\|_2^2 + \mu^2 |\operatorname{tr}(A)|^2}.$$

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Theorem (B. ; 2023)

$\|A\|_{\boldsymbol{X},2}$ is submultiplicative for any choice of distribution.

The main result

- $f(\alpha x) = |\alpha| f(x)$ • $f(x+y) \leq f(x) + f(y)$ • $f(x) = 0 \iff x = 0$ • $f(xy) \leq f(x)f(y)$
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Theorem (B. ; 2023)

$\|A\|_{\boldsymbol{X},d}$ is submultiplicative for any $d \geq 1$ and any choice of distribution.

Sketch of proof (Part 1)

- $f(\alpha x) = |\alpha| f(x)$
- $f(x+y) \leq f(x) + f(y)$
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Theorem

Let $N(\cdot)$ be a matrix norm, and suppose that $\|\cdot\|$ is a submultiplicative matrix norm such that

$$C_m\|A\| \leq N(A) \leq C_M\|A\| \quad \text{for all } A \in M_n$$

where C_m and C_M are positive constants. Then $\frac{C_M}{C_m^2} N(\cdot)$ is also a submultiplicative matrix norm.

Sketch of proof (Part 2)

- $f(\alpha x) = |\alpha| f(x)$
- $f(x+y) \leq f(x) + f(y)$
- $f(x) = 0 \iff x = 0$
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- $\|A\|_{\boldsymbol{X},d}^d = \mathbb{E}\big[|\lambda_1 X_1 + \lambda_2 X_2 + \cdots + \lambda_n X_n|^d\big]$
- $\|A\|_{\boldsymbol{X},2}$ is submultiplicative

Theorem (B. ; 2023)

For any $d \geq 1$ and any distribution,

$$C_m \|A\|_{\boldsymbol{X},2} \leq \|A\|_{\boldsymbol{X},d} \leq C_M \|A\|_{\boldsymbol{X},2},$$

where C_m and C_M depends only on d and not on the dimensions n of the matrices nor on the underlying distribution.

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