From Hardy's Inequality to Modern Generalizations

A Historical and Contemporary Perspective

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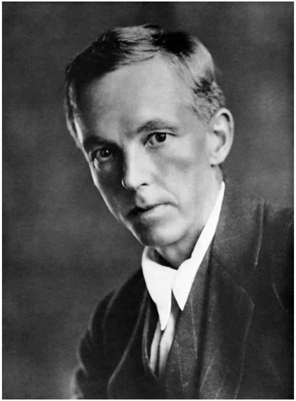
A famous quote



"All analysts spend half their time hunting through the literature for inequalities which they want to use and cannot prove."

– G.H. Hardy

Hilbert's inequality

Theorem (Hilbert ; 1906)

If (a_m) and (b_n) are sequences of positive numbers, then

$$\sum_{m,n=1}^{\infty} \frac{a_m b_n}{m+n} \leq 2\pi \left(\sum_{m=1}^{\infty} a_m^2 \right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} b_n^2 \right)^{\frac{1}{2}}.$$

David Hilbert

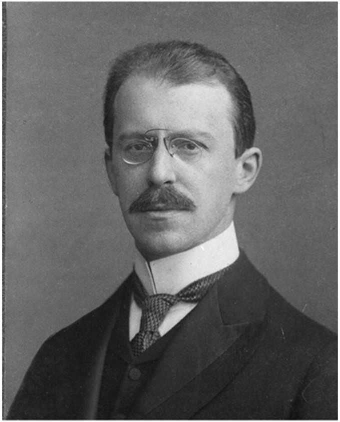
Hilbert's inequality (Optimal constant)

Theorem (Schur ; 1911)

If (a_m) and (b_n) are sequences of positive numbers, then

$$\sum_{m,n=1}^{\infty} \frac{a_m b_n}{m+n} \leq \pi \left(\sum_{m=1}^{\infty} a_m^2 \right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} b_n^2 \right)^{\frac{1}{2}}.$$

Moreover, the constant π is optimal.



Issai Schur

Hilbert's inequality (Generalization)

Theorem (G.H. Hardy, M. Riesz ; 1920)

If (a_m) and (b_n) are sequences of positive numbers and $p \in (1, \infty)$, then

$$\sum_{m,n=1}^{\infty} \frac{a_m b_n}{m+n} \leq \frac{\pi}{\sin(\frac{\pi}{p})} \left(\sum_{m=1}^{\infty} a_m^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} b_n^q \right)^{\frac{1}{q}}.$$

Moreover, the constant $\pi \operatorname{cosec}(\frac{\pi}{p})$ is optimal.

Marcel Riesz

Hardy's inequality

Theorem (Hardy, Riesz ; 1920)

If (a_n) is a sequence of positive numbers and $p \in (1, \infty)$, then

$$\sum_{n=1}^{\infty} \left(\frac{a_1 + a_2 + \cdots + a_n}{n} \right)^p \leq \left(\frac{p^2}{p-1} \right)^p \sum_{n=1}^{\infty} a_n^p.$$

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Question : Is the constant $\left(\frac{p^2}{p-1} \right)^p$ optimal ?

Hardy's inequality (Optimal constant)

Theorem (E. Landau ; 1926)

If (a_n) is a sequence of positive numbers and $p \in (1, \infty)$, then

$$\sum_{n=1}^{\infty} \left(\frac{a_1 + a_2 + \cdots + a_n}{n} \right)^p \leq \left(\frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} a_n^p.$$

Moreover, the constant $\left(\frac{p}{p-1}\right)^p$ is optimal.

Edmund Landau

My supervisors and I

Javad Mashreghi

Frédéric Morneau-Guérin

My cat and I

The main goal

Hardy's inequality

$$\sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} 1 \cdot a_k \right|^p \leq C \sum_{n=1}^{\infty} a_n^p.$$

The main goal

Our inequality

$$\sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k \in \mathcal{N}_n} 1 \cdot a_k \right|^p \leq C \sum_{n=1}^{\infty} a_n^p.$$

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Our inequality

$$\sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k \in \mathcal{N}_n} m_k a_k \right|^p \leq C \sum_{n=1}^{\infty} a_n^p.$$

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$$\sum_{n=1}^{\infty} \left| \frac{1}{M_n} \sum_{k \in \mathcal{N}_n} m_k a_k \right|^p \leq C \sum_{n=1}^{\infty} a_n^p.$$

Definitions

Summation indices

- Let $\mathbb{N} = N_1 \cup N_2 \cup \cdots$ be a partition of $\mathbb{N}$.

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Example

Let $N_1 = \{1, 2, 3\}$ and $N_n = \{2^{n-1} + 1, 2^{n-1} + 2, \ldots, 2^n\}$ if $n \geq 2$. Then

$$\mathcal{N}_1 = \{1, 2, 3\}, \quad \mathcal{N}_n = \{1, 2, \ldots, 2^n\} \quad \text{if } n \geq 2.$$

Definitions

Weights

• $\mathbb{N} = N_1 \cup N_2 \cup \cdots$ • $\mathcal{N}_n := N_1 \cup N_2 \cup \cdots \cup N_n$

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- $\rho := \sup_{n \geq 1} \left(w_n \sum_{k=n}^{\infty} \frac{1}{M_k} \right)$.

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Theorem (B., Mashreghi, Morneau-Guérin ; 2023)

Let $(a_n)_{n\geq 1}$ be a sequence of complex numbers, and let $(m_n)_{n\geq 1}$ be a sequence of weights. Define M_n and ρ as above, and assume that ρ is finite. Then

$$\sum_{n=1}^{\infty} \left| \frac{1}{M_n} \sum_{k\in\mathcal{N}_n} m_k a_k \right|^p \leq \rho \sum_{n=1}^{\infty} |a_n|^p.$$

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Theorem (B., Mashreghi, Morneau-Guérin ; 2023)

Let $(a_n)_{n\geq 1}$ be a sequence of complex numbers, and let $(m_n)_{n\geq 1}$ and $(M_n)_{n\geq 1}$ be two sequences of weights. If

$$\rho := \sup_{n\geq 1} \left(w_n \sum_{k=n}^{\infty} \frac{(w_1 + \cdots + w_k)^{p-1}}{M_k^p} \right)$$

is finite, then

$$\sum_{n=1}^{\infty} \left| \frac{1}{M_n} \sum_{k \in \mathcal{N}_n} m_k a_k \right|^p \leq \rho \sum_{n=1}^{\infty} |a_n|^p .$$

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Example

If $N_n = \{n\}$, $m_n = 1$ and $M_n = n^{1+\varepsilon}$ $(\varepsilon > 0)$ for all $n \geq 1$, then

- $\mathcal{N}_n = \{1, 2, \ldots, n\}$;
- $w_n = 1$;
- $\rho = \sup_{n \geq 1} \sum_{k=n}^{\infty} \frac{1}{k^{1+p\varepsilon}} \leq \zeta(1 + p\varepsilon) < \infty$.

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$$\implies \sum_{n=1}^{\infty} \left| \frac{a_1 + \cdots + a_n}{n^{1+\varepsilon}} \right|^p \leq \zeta(1 + p\varepsilon) \sum_{n=1}^{\infty} |a_n|^p.$$

Lacunary sequence

Definition

A sequence $(n_k)_{k\geq 1}$ of positive integers satisfy the Hadamard gap condition if there exist some $r > 1$ such that

$$\frac{n_{k+1}}{n_k} \geq r, \qquad (k \geq 1).$$

Whenever this is the case, $(n_k)_{k\geq 1}$ is called a *lacunary sequence of ratio* r.

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Example

If $r > 1$, $(r^k)_{k\geq 1}$ is a lacunary sequence of ratio r.

Corollaries

An application to lacunary sequences

Corollary (B., Mashreghi, Morneau-Guérin ; 2023)

Let $(a_n)_{n\geq 1}$ be a sequence of complex numbers, and let $(n_k)_{k\geq 1}$ be a lacunary sequence of ratio r. If $p \in (1, \infty)$, then

$$\sum_{k=1}^{\infty} \left| \frac{1}{n_k^{1/q}} \sum_{j=1}^{n_k} a_j \right|^p \leq \left(\frac{r^{1/q}}{r^{1/q} - 1} \right)^p \sum_{n=1}^{\infty} |a_n|^p.$$

Corollaries

An extreme case : Geometric sequences

Theorem (B., Mashreghi, Morneau-Guérin ; 2023)

Let $(a_n)_{n\geq 1}$ be a sequence of complex numbers and let $b \geq 2$ be an integer. Then

$$\sum_{k=1}^{\infty} \frac{1}{b^k} \left| \sum_{j=1}^{b^k} a_j \right|^2 \leq \frac{\sqrt{b}+1}{\sqrt{b}-1} \sum_{n=1}^{\infty} |a_n|^2.$$

Moreover, the constant $\frac{\sqrt{b}+1}{\sqrt{b}-1}$ is optimal and the above inequality is strict, except if $(a_n)_{n\geq 1}$ is the null sequence.

Corollaries

An extreme case : Geometric sequences

Example

If $b = 4$, then

$$\sum_{n=1}^{\infty} \frac{1}{4^n} \left| \sum_{k=1}^{4^n} a_k \right|^2 \leq 3 \sum_{n=1}^{\infty} |a_n|^2.$$

Moreover, the constant 3 is optimal.

References

- Ludovick Bouthat, Javad Mashreghi & Frédéric Morneau-Guérin (2023) Weighted averages of ℓ^p sequences. *Journal of Mathematical Inequalities.*, Submitted.
- Ludovick Bouthat & Javad Mashreghi (2021) *L*-matrices with lacunary coefficients, *Operators and Matrices*, 15(3) :1045–1053.
- Ludovick Bouthat & Javad Mashreghi (2021) The norm of an infinite *L*-matrix, *Operators and Matrices*, 15(1) :47–58.