The Submultiplicativity of Matrix Norms Induced by Random Vectors

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WCLAM; May 26, 2024

Acknowledgement

This research is a collaborate effort with Pr. Stephan Ramon Garcia and Pr. Ángel Chávez.

Stephan Ramon Garcia

Ángel Chávez

My cat and I

Acknowledgment

This research was done with the financial help of the Vanier Scholarship.

Bourses d’études supérieures du Canada
Vanier
Canada Graduate Scholarships

A new surprising matrix norm

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Then

$$\|A\|_{\boldsymbol{X},d} := \mathbb{E}\Big[|\langle \boldsymbol{X}, \boldsymbol{\lambda}\rangle|^d\Big]^{\frac{1}{d}} = \mathbb{E}\Big[|\lambda_1 X_1 + \lambda_2 X_2 + \cdots + \lambda_n X_n|^d\Big]^{\frac{1}{d}}$$

are matrix norms on the space of Hermitian matrices.

An interesting extension

- $\|A\|_{\boldsymbol{X},d}^d = \mathbb{E}\big[|\lambda_1 X_1 + \lambda_2 X_2 + \cdots + \lambda_n X_n|^d\big]$

Proposition (Aguilar, Chávez, Garcia, Volčič, 2022 ; B. , 2024)

The function

$$|||Z|||_{\boldsymbol{X},d} = \left(\frac{1}{2\pi\binom{d}{d/2}} \int_0^{2\pi} \|e^{it}Z + e^{-it}Z^*\|_{\boldsymbol{X},d}^d \,\mathrm{d}t \right)^{1/d}.$$

defines a norm on $M_n(\mathbb{C})$ which restricts to $\|\cdot\|_{\boldsymbol{X},d}$ on the space of Hermitian matrices.

Some properties

- $\|A\|_{\boldsymbol{X},d}^{d} = \mathbb{E}\big[|\lambda_1 X_1 + \lambda_2 X_2 + \cdots + \lambda_n X_n|^d\big]$
- $|\!|\!| Z |\!|\!|_{\boldsymbol{X},d}^{d} = \frac{1}{2\pi}\binom{d}{d/2}^{-1} \int_0^{2\pi} \|e^{it}Z + e^{-it}Z^*\|_{\boldsymbol{X},d}^{d}\,\mathrm{d}t$

Theorem (B., Chávez, Garcia, Hurley ; 2023)

Under the previous hypothesis,

1. $|\!|\!| UZU^* |\!|\!|_{\boldsymbol{X},d} = |\!|\!| Z |\!|\!|_{\boldsymbol{X},d}$ *for any unitary matrix* U *;*

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3. $\|\!|Z|\!\|_{\boldsymbol{X},d_1} \leq \|\!|Z|\!\|_{\boldsymbol{X},d_2}$ *if* $d_1 \leq d_2$ *;*

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4. $\|Z_1\|_{\boldsymbol{X},d} \leq \|Z_2\|_{\boldsymbol{X},d}$ *if* $\boldsymbol{\lambda}(Z_1)$ *is majorized by* $\boldsymbol{\lambda}(Z_2)$ *;*

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5. $\|Z\|_{\boldsymbol{X},d}$ *is maybe submultiplicative... ?*

Normal random variables

- $\|A\|_{\boldsymbol{X},d}^d = \mathbb{E}\left[|\lambda_1 X_1 + \lambda_2 X_2 + \cdots + \lambda_n X_n|^d\right]$
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Example ($d = 4$)

$$\|\!|Z|\!\|_{\boldsymbol{X},4}^4 = \mu^4(\operatorname{tr} Z)^2(\operatorname{tr} Z^*)^2 + \mu^2\sigma^2\operatorname{tr}(Z^*)^2\operatorname{tr}(Z^2) + \mu^2\sigma^2(\operatorname{tr} Z)^2\operatorname{tr}(Z^{*2}) + 4\mu^2\sigma^2(\operatorname{tr} Z)(\operatorname{tr} Z^*)(\operatorname{tr} Z^*Z) + 2\sigma^4(\operatorname{tr} Z^*Z)^2 + \sigma^4\operatorname{tr}(Z^2)\operatorname{tr}(Z^{*2}).$$

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Example (A is Hermitian)

$$\|A\|_{\boldsymbol{X},d} = \sqrt{2}\sigma\|A\|_{\mathrm{F}}\left(\frac{1}{\sqrt{\pi}}\Gamma\big(\tfrac{d+1}{2}\big)\,{}_1F_1\Big(-\frac{d}{2};\,\frac{1}{2};\,-\frac{\mu^2(\operatorname{tr} A)^2}{2\sigma^2\|A\|_{\mathrm{F}}^2}\Big)\right)^{1/d},$$

where ${}_1F_1(\alpha;\beta;z)$ is Kummer's confluent hypergeometric function.

Normal random variables

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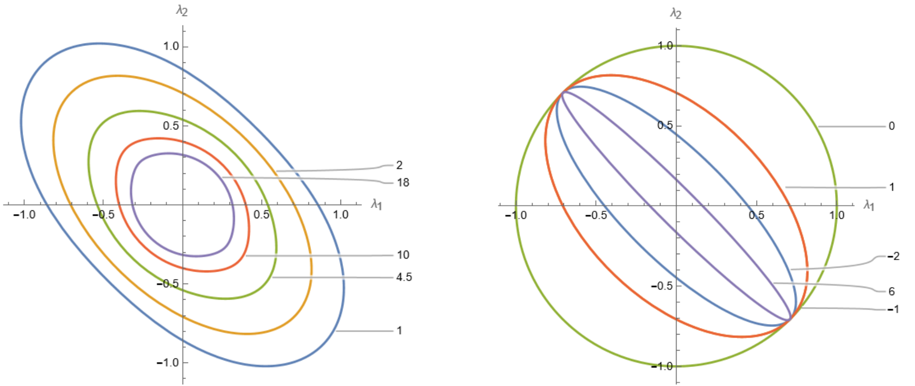

(Left) Unit circles for $\|\cdot\|_{\boldsymbol{X},d}$ with $d = 1, 2, 4.5, 10, 18$, in which X_1, X_2 are normal random variables with $\mu = \sigma = 1$. (Right) Unit circles for $\|\cdot\|_{\boldsymbol{X},10}$, in which X_1, X_2 are normal random variables with $\mu = -2, -1, 0, 1, 6$ and variance $\sigma^2 = 1$.

Uniform and exponential random variable

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Example (Uniform random variable on $(-1,1)$; $d = 6$; A is Hermitian)

$$\|A\|_{\boldsymbol{X},6}^6 = \frac{1}{63}\big(35(\operatorname{tr} A^2)^3 - 42\operatorname{tr}(A^4)\operatorname{tr}(A^2) + 16\operatorname{tr}(A^6)\big).$$

Uniform and exponential random variable

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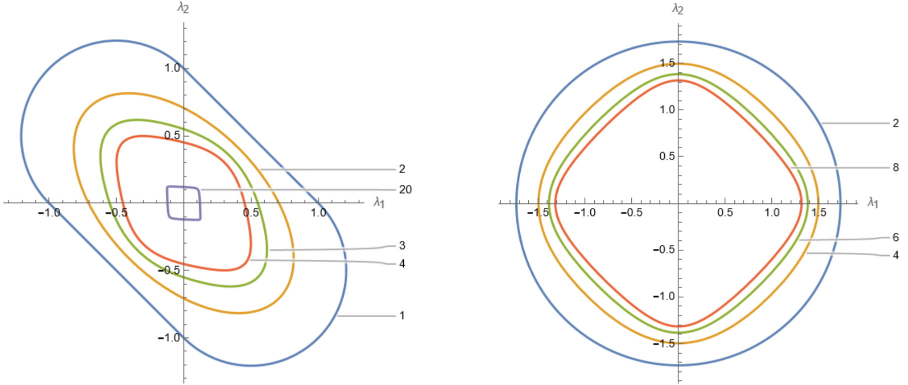
Example (d is even; A is Hermitian)

$$\|A\|_{\boldsymbol{X},d}^d = d!\, h_d(\lambda_1, \lambda_2, \ldots, \lambda_n) = d! \sum_{1\leq k_1\leq\cdots\leq k_d\leq n} \lambda_{k_1}\lambda_{k_2}\cdots\lambda_{k_d},$$

where h_d is the *complete homogeneous symmetric polynomial* of degree d.

More figures

- $\|A\|_{\mathbf{X},d}^d = \mathbb{E}\big[|\lambda_1 X_1 + \lambda_2 X_2 + \cdots + \lambda_n X_n|^d\big]$
- $\|Z\|_{\mathbf{X},d}^d = \frac{1}{2\pi}\binom{d}{d/2}^{-1}\int_0^{2\pi}\|e^{it}Z + e^{-it}Z^*\|_{\mathbf{X},d}^d\,\mathrm{d}t$



(Left) Unit circles for $\|\cdot\|_{\mathbf{X},d}$ with $d = 1, 2, 3, 4, 20$, in which X_1 and X_2 are exponential random variables. (Right) Unit circles for $\|\cdot\|_{\mathbf{X},d}$ with $d = 2, 4, 6, 8$, in which X_1 and X_2 are Uniform random variables on $[-1, 1]$.

Spectral graph theory

- $\|A\|_{\boldsymbol{X},d}^d = \mathbb{E}\big[|\lambda_1 X_1 + \lambda_2 X_2 + \cdots + \lambda_n X_n|^d\big]$
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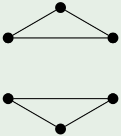
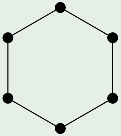
Example

Let $K := \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, and let $X_i \sim \Gamma(1, 1/2)$. Then



$$\mapsto \; A = \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix},\; \sigma(A) = \left\{ \begin{matrix} -1, -1, -1, \\ -1, \;\; 2, \;\; 2 \end{matrix} \right\} \;\Rightarrow\; \begin{matrix} \|A\|_{\mathsf{F}} = 2\sqrt{3} \\ \|A\|_{\boldsymbol{X},6}^6 = 1350 \end{matrix};$$



$$\mapsto \; A = \begin{bmatrix} 0 & K \\ K & 0 \end{bmatrix},\; \sigma(A) = \left\{ \begin{matrix} -1, -1, \;\; 1, \\ 1, \;\; 2, -2 \end{matrix} \right\} \;\Rightarrow\; \begin{matrix} \|A\|_{\mathsf{F}} = 2\sqrt{3} \\ \|A\|_{\boldsymbol{X},6}^6 = 1260 \end{matrix}.$$

A small rant...

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- The only difference is the existence of matrix multiplication in M_n.

$\implies$ Matrix norms which are not *submultiplicative* are only vector norms *disguised* as matrix norms.

Definition

A function $f : M_n \to \mathbb{R}$ is *submultiplicative* if for any $X, Y \in M_n$,

$$f(XY) \leq f(X)f(Y).$$

What about $\|\!| \cdot \|\!|_{\boldsymbol{X},d}$?

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Main question

Under which conditions on the distribution underlying the random vector $\boldsymbol{X}$ is the matrix norm $\|\!| \cdot \|\!|_{\boldsymbol{X},d}$ submultiplicative?

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Under which conditions on the distribution underlying the random vector $\boldsymbol{X}$ *is the matrix norm* $|||\cdot|||_{\boldsymbol{X},d}$ *submultiplicative?*

Remark

The same question on $\|\cdot\|_{\boldsymbol{X},d}$, although much simpler, is ill-defined since the set of Hermitian matrices is not closed under matrix multiplication.

A small rant... *Part 2*

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- If $\|\cdot\|$ is a norm, then $\gamma\|\cdot\|$ is a norm for any $\gamma > 0$ with essentially the same geometry and properties as the original norm.

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- For $\gamma > 0$ large enough, $\gamma\|AB\| \leq (\gamma\|A\|)(\gamma\|B\|)$ for all $A, B \in M_n$.

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$\implies$ Every matrix norm is submultiplicative, up to scalar multiplication.

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$\implies$ Every matrix norm is submultiplicative, up to scalar multiplication.

- In several context, it is desirable that $\gamma > 0$ can be chosen to be independent of the dimension n of the matrices. This yield a *single* submultiplicative matrix norm instead of a *family* of submultiplicative matrix norm.

What about $|||\cdot|||_{\boldsymbol{X},d}$?

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Main question (Correct version)

Under which conditions on the distribution underlying the random vector $\boldsymbol{X}$ *does there exist a constant* $\gamma > 0$ *independent of* n *such that the matrix norm* $|||\cdot|||_{\boldsymbol{X},d}$ *is submultiplicative?*

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Main question (Correct version)

Under which conditions on the distribution underlying the random vector $\boldsymbol{X}$ does there exist a constant $\gamma > 0$ independent of n such that the matrix norm $\|\cdot\|_{\boldsymbol{X},d}$ is submultiplicative?

Remark

If such a γ exists, then one can consider the random vector $\gamma\boldsymbol{X}$. It follows that $\|\cdot\|_{\gamma\boldsymbol{X},d} = \gamma\|\cdot\|_{\boldsymbol{X},d}$ is submultiplicative.

The main result

- $\|A\|_{\boldsymbol{X},d}^d = \mathbb{E}\left[|\lambda_1 X_1 + \lambda_2 X_2 + \cdots + \lambda_n X_n|^d\right]$
- $\|\!|Z|\!\|_{\boldsymbol{X},d}^d = \frac{1}{2\pi}\binom{d}{d/2}^{-1}\int_0^{2\pi} \|e^{it}Z + e^{-it}Z^*\|_{\boldsymbol{X},d}^d \,\mathrm{d}t$

Theorem (B. , 2024)

Let $d \geq 1$ and $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$, where $X_1, X_2, \ldots, X_n \in L^p(\Omega, \mathcal{F}, \boldsymbol{P})$ are iid random variables and $p = \max\{d, \eta\}$ for some $\eta > 2$. Then there exists a constant $\gamma_d > 0$, independent of n, such that $\gamma_d \|\!|Z|\!\|_{\boldsymbol{X},d}$ is a submultiplicative matrix norm on M_n.

In particular, there exists a constant $\gamma_d > 0$, independent of n, such that $\gamma_d \|\!|Z|\!\|_{\boldsymbol{X},d}$ is a submultiplicative matrix norm on M_n for any $d \geq 2$.

The case $d = 2$

- $\|A\|_{\boldsymbol{X},d}^d = \mathbb{E}\big[|\lambda_1 X_1 + \lambda_2 X_2 + \cdots + \lambda_n X_n|^d\big]$
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- If $d = 2$, μ is the mean, and σ is the standard deviation of the distribution of $X_1, X_2, \ldots, X_n \in L^2(\Omega, \mathcal{F}, \boldsymbol{P})$, then

$$\|\!|Z|\!\|_{\boldsymbol{X},2} = \sqrt{\sigma^2\|Z\|_{\mathrm{F}}^2 + \mu^2|\operatorname{tr}(Z)|^2}.$$

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- $\|Z\|_{\boldsymbol{X},d}^d = \frac{1}{2\pi}\binom{d}{d/2}^{-1}\int_0^{2\pi}\|e^{it}Z + e^{-it}Z^*\|_{\boldsymbol{X},d}^d \,\mathrm{d}t$

- If $d = 2$, μ is the mean, and σ is the standard deviation of the distribution of $X_1, X_2, \ldots, X_n \in L^2(\Omega, \mathcal{F}, \boldsymbol{P})$, then

$$\|Z\|_{\boldsymbol{X},2} = \sqrt{\sigma^2\|Z\|_{\mathrm{F}}^2 + \mu^2|\operatorname{tr}(Z)|^2}.$$

Theorem (B. , 2024)

Let $d = 2$ and $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ where $X_1, X_2, \ldots, X_n \in L^d(\Omega, \mathcal{F}, \boldsymbol{P})$ are iid random variables. Then $\frac{\sqrt{\sigma^2+\mu^2}}{\sigma^2}\|Z\|_{\boldsymbol{X},2}$ is a submultiplicative matrix norm on M_n. Moreover, the constant $\frac{\sqrt{\sigma^2+\mu^2}}{\sigma^2}$ is optimal.

Sketch of proof (Part 1)

- $\|A\|_{\boldsymbol{X},d}^{d} = \mathbb{E}\big[|\lambda_1 X_1 + \lambda_2 X_2 + \cdots + \lambda_n X_n|^d\big]$
- $\|Z\|_{\boldsymbol{X},d}^{d} = \frac{1}{2\pi}\binom{d}{d/2}^{-1}\int_0^{2\pi}\|e^{it}Z + e^{-it}Z^*\|_{\boldsymbol{X},d}^{d}\,\mathrm{d}t$

Theorem (Folklore)

Let $N(\cdot)$ be a matrix norm, and suppose that $\|\cdot\|$ is a submultiplicative matrix norm such that

$$C_m\|A\| \leq N(A) \leq C_M\|A\| \quad \text{for all } A \in M_n$$

where C_m and C_M are positive constants. Then $\frac{C_M}{C_m^2}N(\cdot)$ is also a submultiplicative matrix norm.

Sketch of proof (Part 2)

- $\|A\|_{\boldsymbol{X},d}^d = \mathbb{E}\left[|\lambda_1 X_1 + \lambda_2 X_2 + \cdots + \lambda_n X_n|^d\right]$
- $\|Z\|_{\boldsymbol{X},d}^d = \frac{1}{2\pi}\binom{d}{d/2}^{-1} \int_0^{2\pi} \|e^{it}Z + e^{-it}Z^*\|_{\boldsymbol{X},d}^d \,\mathrm{d}t$

Theorem (B., 2024)

Let $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$, where $X_1, X_2, \ldots, X_n \in L^p(\Omega, \mathcal{F}, \boldsymbol{P})$ are iid random variables of pth standardized absolute moment $\tilde{\mu}_p$.
Then if $p = d \geq 2$,

$$\sqrt{2}\binom{d}{d/2}^{-1/d} \|Z\|_{\boldsymbol{X},2} \leq \|Z\|_{\boldsymbol{X},d} \leq 4\left(\frac{B_d \tilde{\mu}_d}{2\binom{d}{d/2}}\right)^{1/d} \|Z\|_{\boldsymbol{X},2},$$

and if $1 \leq d \leq 2$ and $p > 2$,

$$4\left(\frac{(2B_p\tilde{\mu}_p)^{\frac{d-2}{p-2}}}{8\binom{d}{d/2}}\right)^{1/d} \|Z\|_{\boldsymbol{X},2} \leq \|Z\|_{\boldsymbol{X},d} \leq \sqrt{2}\binom{d}{d/2}^{-1/d} \|Z\|_{\boldsymbol{X},2},$$

where B_p is the constant in the Marcinkiewicz–Zygmund inequality.

Open questions

1. For $1 \leq d < 2$, does there exists a constant $\gamma_d > 0$, independent of n, such that $\gamma_d \| Z \|_{\boldsymbol{X},d}$ is a submultiplicative matrix norm, even if $X_i \notin L^p(\Omega, \mathcal{F}, \boldsymbol{P})$ for $p > 2$?

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4. Can we characterize the norms $\|\cdot\|_{\boldsymbol{X},d}$ that arise from an inner product ?

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