Matrix Norms Induced by Random Vectors

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Stephan Ramon Garcia

Ángel Chávez

My cat and I

Acknowledgment

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A new surprising matrix norm

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Then

$$\|A\|_{\boldsymbol{X},d} := \mathbb{E}\Big[|\langle \boldsymbol{X}, \boldsymbol{\lambda}\rangle|^d\Big]^{\frac{1}{d}} = \mathbb{E}\Big[|\lambda_1 X_1 + \lambda_2 X_2 + \cdots + \lambda_n X_n|^d\Big]^{\frac{1}{d}}$$

are matrix norms on the space of Hermitian matrices.

An interesting extension

- $\|A\|_{\boldsymbol{X},d}^d = \mathbb{E}\big[|\lambda_1 X_1 + \lambda_2 X_2 + \cdots + \lambda_n X_n|^d\big]$

Proposition (Aguilar, Chávez, Garcia, Volčič, 2022 ; B. , 2024)

The function

$$|\!|\!| Z |\!|\!|_{\boldsymbol{X},d} = \left(\frac{1}{2\pi\binom{d}{d/2}} \int_0^{2\pi} \|e^{it}Z + e^{-it}Z^*\|_{\boldsymbol{X},d}^d \, \mathrm{d}t \right)^{1/d}.$$

defines a norm on $M_n(\mathbb{C})$ which restricts to $\|\cdot\|_{\boldsymbol{X},d}$ on the space of Hermitian matrices.

Some properties

- $\|A\|_{\boldsymbol{X},d}^d = \mathbb{E}\big[|\lambda_1 X_1 + \lambda_2 X_2 + \cdots + \lambda_n X_n|^d\big]$
- $|\!|\!| Z |\!|\!|_{\boldsymbol{X},d}^d = \frac{1}{2\pi}\binom{d}{d/2}^{-1} \int_0^{2\pi} \|e^{it}Z + e^{-it}Z^*\|_{\boldsymbol{X},d}^d \,\mathrm{d}t$

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Under the previous hypothesis,

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4. $\|Z_1\|_{\mathbf{X},d} \leq \|Z_2\|_{\mathbf{X},d}$ *if* $\boldsymbol{\lambda}(Z_1)$ *is majorized by* $\boldsymbol{\lambda}(Z_2)$ *;*

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4. $\|Z_1\|_{\boldsymbol{X},d} \leq \|Z_2\|_{\boldsymbol{X},d}$ *if* $\boldsymbol{\lambda}(Z_1)$ *is majorized by* $\boldsymbol{\lambda}(Z_2)$;
5. $\|Z\|_{\boldsymbol{X},d}$ *is maybe submultiplicative... ?*

Normal random variables

- $\|A\|_{\boldsymbol{X},d}^d = \mathbb{E}\big[|\lambda_1 X_1 + \lambda_2 X_2 + \cdots + \lambda_n X_n|^d\big]$
- $|\!|\!|Z|\!|\!|_{\boldsymbol{X},d}^d = \frac{1}{2\pi}\binom{d}{d/2}^{-1}\int_0^{2\pi}\|e^{it}Z + e^{-it}Z^*\|_{\boldsymbol{X},d}^d \,\mathrm{d}t$

Example ($d = 4$)

$$|\!|\!|Z|\!|\!|_{\boldsymbol{X},4}^4 = \mu^4(\operatorname{tr} Z)^2(\operatorname{tr} Z^*)^2 + \mu^2\sigma^2 \operatorname{tr}(Z^*)^2 \operatorname{tr}(Z^2) + \mu^2\sigma^2(\operatorname{tr} Z)^2 \operatorname{tr}(Z^{*2}) + 4\mu^2\sigma^2(\operatorname{tr} Z)(\operatorname{tr} Z^*)(\operatorname{tr} Z^* Z) + 2\sigma^4(\operatorname{tr} Z^* Z)^2 + \sigma^4 \operatorname{tr}(Z^2)\operatorname{tr}(Z^{*2}).$$

Normal random variables

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Example (A is Hermitian)

$$\|A\|_{\boldsymbol{X},d} = \sqrt{2}\sigma\|A\|_{\mathrm{F}}\left(\frac{1}{\sqrt{\pi}}\Gamma\big(\tfrac{d+1}{2}\big)\,{}_1F_1\Big(-\frac{d}{2};\ \frac{1}{2};\ -\frac{\mu^2(\operatorname{tr} A)^2}{2\sigma^2\|A\|_{\mathrm{F}}^2}\Big)\right)^{1/d},$$

where ${}_1F_1(\alpha;\beta;z)$ is Kummer's confluent hypergeometric function.

Normal random variables

- $\|A\|_{\boldsymbol{X},d}^{d} = \mathbb{E}\big[|\lambda_1 X_1 + \lambda_2 X_2 + \cdots + \lambda_n X_n|^d\big]$
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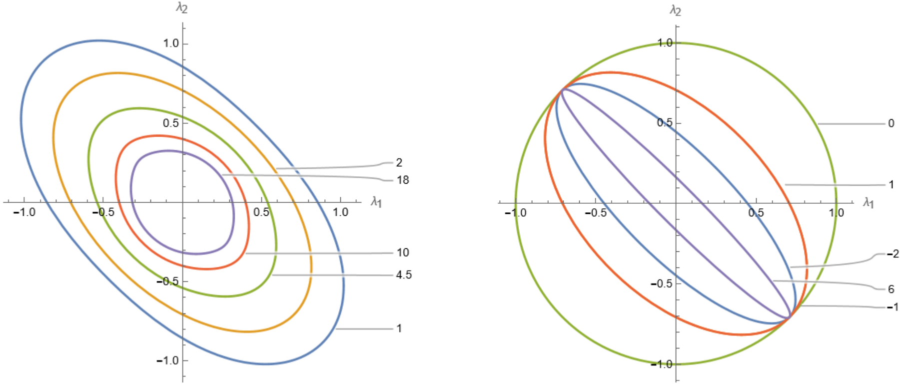

(Left) Unit circles for $\|\cdot\|_{\boldsymbol{X},d}$ with $d = 1, 2, 4.5, 10, 18$, in which X_1, X_2 are normal random variables with $\mu = \sigma = 1$. (Right) Unit circles for $\|\cdot\|_{\boldsymbol{X},10}$, in which X_1, X_2 are normal random variables with $\mu = -2, -1, 0, 1, 6$ and variance $\sigma^2 = 1$.

Uniform and exponential random variable

- $\|A\|_{\boldsymbol{X},d}^d = \mathbb{E}\big[|\lambda_1 X_1 + \lambda_2 X_2 + \cdots + \lambda_n X_n|^d\big]$
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Example (Uniform random variable on $(-1,1)$; $d=6$; A is Hermitian)

$$\|A\|_{\boldsymbol{X},6}^6 = \frac{1}{63}\big(35(\operatorname{tr} A^2)^3 - 42\operatorname{tr}(A^4)\operatorname{tr}(A^2) + 16\operatorname{tr}(A^6)\big).$$

Uniform and exponential random variable

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Example (d is even ; A is Hermitian)

$$\|A\|_{\boldsymbol{X},d}^{d} = d!\, h_d(\lambda_1, \lambda_2, \ldots, \lambda_n) = d! \sum_{1 \le k_1 \le \cdots \le k_d \le n} \lambda_{k_1} \lambda_{k_2} \cdots \lambda_{k_d},$$

where h_d is the *complete homogeneous symmetric polynomial* of degree d.

More figures

- $\|A\|_{\mathbf{X},d}^d = \mathbb{E}\big[|\lambda_1 X_1 + \lambda_2 X_2 + \cdots + \lambda_n X_n|^d\big]$
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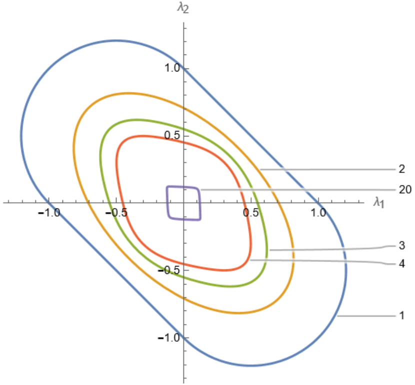

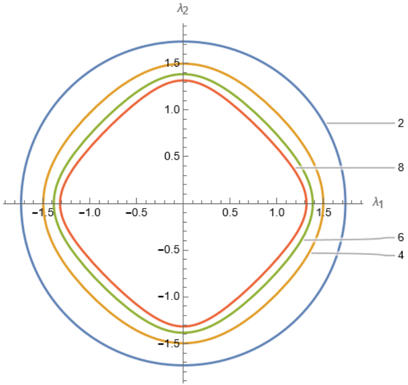

(Left) Unit circles for $\|\cdot\|_{\mathbf{X},d}$ with $d = 1, 2, 3, 4, 20$, in which X_1 and X_2 are exponential random variables. (Right) Unit circles for $\|\cdot\|_{\mathbf{X},d}$ with $d = 2, 4, 6, 8$, in which X_1 and X_2 are Uniform random variables on $[-1, 1]$.

Spectral graph theory

- $\|A\|_{\boldsymbol{X},d}^d = \mathbb{E}\big[|\lambda_1 X_1 + \lambda_2 X_2 + \cdots + \lambda_n X_n|^d\big]$
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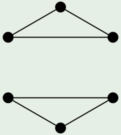
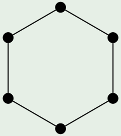
Example

Let $K := \begin{bmatrix} 0&1&1\\1&0&1\\1&1&0\end{bmatrix}$, and let $X_i \sim \Gamma(1, 1/2)$. Then



$$\mapsto \; A = \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix},\ \sigma(A) = \left\{\begin{matrix} -1, & -1, & -1, \\ -1, & 2, & 2 \end{matrix}\right\} \Rightarrow \begin{matrix} \|A\|_{\mathrm{F}} = 2\sqrt{3} \\ \|A\|_{\mathbf{X},6}^6 = 1350 \end{matrix};$$



$$\mapsto \; A = \begin{bmatrix} 0 & K \\ K & 0 \end{bmatrix},\ \sigma(A) = \left\{\begin{matrix} -1, & -1, & 1, \\ 1, & 2, & -2 \end{matrix}\right\} \Rightarrow \begin{matrix} \|A\|_{\mathrm{F}} = 2\sqrt{3} \\ \|A\|_{\mathbf{X},6}^6 = 1260 \end{matrix}.$$

A small rant...

• $\|A\|_{\boldsymbol{X},d}^{d} = \mathbb{E}\left[|\lambda_1 X_1 + \lambda_2 X_2 + \cdots + \lambda_n X_n|^d\right]$ • $\|Z\|_{\boldsymbol{X},d}^{d} = \frac{1}{2\pi}\binom{d}{d/2}^{-1}\int_0^{2\pi}\|e^{it}Z + e^{-it}Z^*\|_{\boldsymbol{X},d}^{d}\,\mathrm{d}t$

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- The only difference is the existence of matrix multiplication in M_n.

$\implies$ Matrix norms which are not *submultiplicative* are only vector norms *disguised* as matrix norms.

Definition

A function $f : M_n \to \mathbb{R}$ is *submultiplicative* if for any $X, Y \in M_n$,

$$f(XY) \leq f(X)f(Y).$$

What about $\|\!|\cdot|\!\|_{\boldsymbol{X},d}$?

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Main question

Under which conditions on the distribution underlying the random vector $\boldsymbol{X}$ *is the matrix norm* $\|\!|\cdot|\!\|_{\boldsymbol{X},d}$ *submultiplicative ?*

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Remark

The same question on $\|\cdot\|_{\boldsymbol{X},d}$, although much simpler, is ill-defined since the set of Hermitian matrices is not closed under matrix multiplication.

A small rant... *Part 2*

- $\|A\|_{\boldsymbol{X},d}^{d} = \mathbb{E}\big[|\lambda_1 X_1 + \lambda_2 X_2 + \cdots + \lambda_n X_n|^d\big]$
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- If $\|\cdot\|$ is a norm, then $\gamma\|\cdot\|$ is a norm for any $\gamma > 0$ with essentially the same geometry and properties as the original norm.

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- For $\gamma > 0$ large enough, $\gamma\|AB\| \leq (\gamma\|A\|)(\gamma\|B\|)$ for all $A, B \in M_n$.

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$\implies$ Every matrix norm is submultiplicative, up to scalar multiplication.

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- In several context, it is desirable that $\gamma > 0$ can be chosen to be independent of the dimension n of the matrices. This yield a *single* submultiplicative matrix norm instead of a *family* of submultiplicative matrix norm.

What about $\lvert\lvert\lvert \cdot \rvert\rvert\rvert_{\boldsymbol{X},d}$?

- $\|A\|_{\boldsymbol{X},d}^d = \mathbb{E}\big[|\lambda_1 X_1 + \lambda_2 X_2 + \cdots + \lambda_n X_n|^d\big]$
- $\lvert\lvert\lvert Z \rvert\rvert\rvert_{\boldsymbol{X},d}^d = \frac{1}{2\pi}\binom{d}{d/2}^{-1} \int_0^{2\pi} \|e^{it}Z + e^{-it}Z^*\|_{\boldsymbol{X},d}^d \, \mathrm{d}t$

Main question (Correct version)

Under which conditions on the distribution underlying the random vector $\boldsymbol{X}$ does there exist a constant $\gamma > 0$ independent of n such that the matrix norm $\lvert\lvert\lvert \cdot \rvert\rvert\rvert_{\boldsymbol{X},d}$ is submultiplicative?

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Under which conditions on the distribution underlying the random vector $\boldsymbol{X}$ does there exist a constant $\gamma > 0$ independent of n such that the matrix norm $\|\cdot\|_{\boldsymbol{X},d}$ is submultiplicative?

Remark

If such a γ exists, then one can consider the random vector $\gamma\boldsymbol{X}$. It follows that $\|\cdot\|_{\gamma\boldsymbol{X},d} = \gamma\|\cdot\|_{\boldsymbol{X},d}$ is submultiplicative.

The main result

- $\|A\|_{\boldsymbol{X},d}^d = \mathbb{E}\big[|\lambda_1 X_1 + \lambda_2 X_2 + \cdots + \lambda_n X_n|^d\big]$
- $\|\!|Z|\!\|_{\boldsymbol{X},d}^d = \frac{1}{2\pi}\binom{d}{d/2}^{-1}\int_0^{2\pi} \|e^{it}Z + e^{-it}Z^*\|_{\boldsymbol{X},d}^d \,\mathrm{d}t$

Theorem (B. , 2024)

Let $d \geq 1$ and $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$, where $X_1, X_2, \ldots, X_n \in L^p(\Omega, \mathcal{F}, \boldsymbol{P})$ are iid random variables and $p = \max\{d, \eta\}$ for some $\eta > 2$. Then there exists a constant $\gamma_d > 0$, independent of n, such that $\gamma_d \|\!|Z|\!\|_{\boldsymbol{X},d}$ is a submultiplicative matrix norm on M_n.

In particular, there exists a constant $\gamma_d > 0$, independent of n, such that $\gamma_d \|\!|Z|\!\|_{\boldsymbol{X},d}$ is a submultiplicative matrix norm on M_n for any $d \geq 2$.

The case $d = 2$

- $\|A\|_{\boldsymbol{X},d}^d = \mathbb{E}\big[|\lambda_1 X_1 + \lambda_2 X_2 + \cdots + \lambda_n X_n|^d\big]$
- $\|\!|Z|\!\|_{\boldsymbol{X},d}^d = \frac{1}{2\pi}\binom{d}{d/2}^{-1}\int_0^{2\pi}\|e^{it}Z + e^{-it}Z^*\|_{\boldsymbol{X},d}^d\,\mathrm{d}t$

- If $d = 2$, μ is the mean, and σ is the standard deviation of the distribution of $X_1, X_2, \ldots, X_n \in L^2(\Omega, \mathcal{F}, \boldsymbol{P})$, then

$$\|\!|Z|\!\|_{\boldsymbol{X},2} = \sqrt{\sigma^2\|Z\|_{\mathrm{F}}^2 + \mu^2|\operatorname{tr}(Z)|^2}.$$

The case $d = 2$

- $\|A\|_{\boldsymbol{X},d}^d = \mathbb{E}\left[|\lambda_1 X_1 + \lambda_2 X_2 + \cdots + \lambda_n X_n|^d\right]$
- $\|Z\|_{\boldsymbol{X},d}^d = \frac{1}{2\pi}\binom{d}{d/2}^{-1}\int_0^{2\pi}\|e^{it}Z + e^{-it}Z^*\|_{\boldsymbol{X},d}^d \,\mathrm{d}t$

- If $d = 2$, μ is the mean, and σ is the standard deviation of the distribution of $X_1, X_2, \ldots, X_n \in L^2(\Omega, \mathcal{F}, \boldsymbol{P})$, then

$$\|Z\|_{\boldsymbol{X},2} = \sqrt{\sigma^2\|Z\|_{\mathrm{F}}^2 + \mu^2|\operatorname{tr}(Z)|^2}.$$

Theorem (B. , 2024)

Let $d = 2$ and $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ where $X_1, X_2, \ldots, X_n \in L^d(\Omega, \mathcal{F}, \boldsymbol{P})$ are iid random variables. Then $\frac{\sqrt{\sigma^2+\mu^2}}{\sigma^2}\|Z\|_{\boldsymbol{X},2}$ is a submultiplicative matrix norm on M_n. Moreover, the constant $\frac{\sqrt{\sigma^2+\mu^2}}{\sigma^2}$ is optimal.

Sketch of proof (Part 1)

- $\|A\|_{\boldsymbol{X},d}^d = \mathbb{E}\big[|\lambda_1 X_1 + \lambda_2 X_2 + \cdots + \lambda_n X_n|^d\big]$
- $\|\!|Z|\!\|_{\boldsymbol{X},d}^d = \frac{1}{2\pi}\binom{d}{d/2}^{-1}\int_0^{2\pi} \|e^{it}Z + e^{-it}Z^*\|_{\boldsymbol{X},d}^d \,\mathrm{d}t$

Theorem (Folklore)

Let $N(\cdot)$ be a matrix norm, and suppose that $\|\cdot\|$ is a submultiplicative matrix norm such that

$$C_m\|A\| \le N(A) \le C_M\|A\| \quad \text{for all } A \in M_n$$

where C_m and C_M are positive constants. Then $\frac{C_M}{C_m^2}N(\cdot)$ is also a submultiplicative matrix norm.

Sketch of proof (Part 2)

- $\|A\|_{\boldsymbol{X},d}^d = \mathbb{E}\left[|\lambda_1 X_1 + \lambda_2 X_2 + \cdots + \lambda_n X_n|^d\right]$
- $\|Z\|_{\boldsymbol{X},d}^d = \frac{1}{2\pi}\binom{d}{d/2}^{-1}\int_0^{2\pi}\|e^{it}Z + e^{-it}Z^*\|_{\boldsymbol{X},d}^d \,\mathrm{d}t$

Theorem (B., 2024)

Let $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$, where $X_1, X_2, \ldots, X_n \in L^p(\Omega, \mathcal{F}, \boldsymbol{P})$ are iid random variables of pth standardized absolute moment $\tilde{\mu}_p$.
Then if $p = d \geq 2$,

$$\sqrt{2}\binom{d}{d/2}^{-1/d}\|Z\|_{\boldsymbol{X},2} \leq \|Z\|_{\boldsymbol{X},d} \leq 4\left(\frac{B_d\tilde{\mu}_d}{2\binom{d}{d/2}}\right)^{1/d}\|Z\|_{\boldsymbol{X},2},$$

and if $1 \leq d \leq 2$ and $p > 2$,

$$4\left(\frac{(2B_p\tilde{\mu}_p)^{\frac{d-2}{p-2}}}{8\binom{d}{d/2}}\right)^{1/d}\|Z\|_{\boldsymbol{X},2} \leq \|Z\|_{\boldsymbol{X},d} \leq \sqrt{2}\binom{d}{d/2}^{-1/d}\|Z\|_{\boldsymbol{X},2},$$

where B_p is the constant in the Marcinkiewicz–Zygmund inequality.

Open questions

1. For $1 \leq d < 2$, does there exists a constant $\gamma_d > 0$, independent of n, such that $\gamma_d \|\!|Z|\!\|_{\boldsymbol{X},d}$ is a submultiplicative matrix norm, even if $X_i \notin L^p(\Omega, \mathcal{F}, \boldsymbol{P})$ for $p > 2$?

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4. Can we characterize the norms $\|\cdot\|_{\boldsymbol{X},d}$ that arise from an inner product ?

References

- Bouthat, L. : On the Submultiplicativity of Matrix Norms Induced by Random Vectors. *Complex Anal. Oper. Theory*, 18, 73, 2024.
- Bouthat, L., Chávez, Á., Garcia, S.R. : Hunter's positivity theorem and random vector norms. *IWOTA 2023*, Submitted, 2024.
- Chávez, Á., Garcia, S.R., Hurley, J. : Norms on complex matrices induced by random vectors II : Extension of weakly unitarily invariant norms. *Canadian Mathematical Bulletin*, 1–11, 2023.
- Chávez, Á., Garcia, S.R., Hurley, J. : Norms on complex matrices induced by random vectors. *Canadian Mathematical Bulletin* 66(3), 808–826, 2023.
- Aguilar, K., Chávez, Á., Garcia, S.R. and Volčič, J. : Norms on complex matrices induced by complete homogeneous symmetric polynomials. *Bull. London Math. Soc.*, 54 : 2078–2100, 2022.