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Functions of perturbed commuting dissipative operators. (English)

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The purpose of the paper is to study the behaviour of functions $f(L,M)$ of commuting maximal dissipative operators L and M under perturbation. In particular, given commuting paris (L_1,L_2) and (M_1,M_2) of maximal dissipative operators, the conditions on fonctions f under which the following Lipschitz type inequality holds are studied:

$$\left\| f(L_1,M_1) - f(L_2,M_2) \right\| \le \text{const} \cdot \max\left\{ \|L_2 - L_1\|, \|M_2 - M_1\| \right\}.$$

Recall that a (not necessarily bounded) linear operator L in a Hilbert space $\mathcal{H}$ with domain $\mathcal{D}_L$ is called *dissipative* if $\operatorname{Im}(Lx,x) \ge 0$ for every x in $\mathcal{D}_L$. It is called *maximal dissipative* if it does not have a proper dissipative extension.

The article follows on from previous work by both authors or by the second author alone. Note that the methods developed in these earlier papers mentioned do not work in the case of commuting maximal dissipative operators. Hence the methods and techniques elaborated in the present article, which are more refined, constitute an innovation.

In §2 the authors give a brief introduction in dissipative operators and recall useful definitions such as the *Cayley transform* of a dissipative operator and the *resolvant self-adjoint dilation* of a maximal dissipative operator. Drawing on the Sz.-Nagy-Foias functional calculus, they then explain how to construct a natural functional calculus. Finally, they explain how to define the semi-spectral measure of the Cayley transform of a dissipative operator.

In §3 the authors proceed, given commuting maximal dissipative operators L and M in a Hilbert space $\mathcal{H}$, to define a semi-spectral measure for such a pair of operators and to construct of a functional calculus

$$f \mapsto f(L,M).$$

§4 and §5 provide a brief introduction to the notions and concepts pertaining to the theory of Besov spaces which will be used later in the article as well as the definition of double operator integrals with respect to spectral measures, i.e., expressions of the form

$$\int\int_{\mathcal{X}_1\times\mathcal{X}_2} \Phi(x_1,x_2)\; dE_1(x)QdE_2(y),$$

where E_1 and E_2 are spectral measures on a Hilbert space $\mathcal{H}$, Φ is a bounded measurable function and Q is a bounded linear operator on $\mathcal{H}$.

The purpose of §6 is to give an alternative proof of the main result presented in a 2019 paper by the same pair of authors, which establish (i) an integral formula for the operator difference $f(L) - f(M)$, where L and M are arbitrary maximal dissipative operators with bounded $L - M$ and f belongs to the space of operator Lipschitz functions on $\mathrm{clos}\mathbb{C}_+$ that are analytic in $\mathbb{C}_+$; and (ii) operator Lipschitz estimates for functions of a single dissipative operator.

This new approach in the case of functions of a single maximal dissipative operator uses an idea, which will be used in §9 to obtain the main result of this paper.

In §7 the authors obtain estimates of divided differences in Haagerup tensor products. These estimates will be put to use in the proof of the main result of the paper.

§8 is devoted to functions of noncommuting maximal dissipative operators; this will also be used in §9.

Lastly, in §10 Hölder type estimates and estimates in Schatten p-norms for functions of pairs of commuting dissipative operators under perturbation are deduced from the main result of the previous Section.

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MSC:

46E35	Sobolev spaces and other spaces of "smooth" functions, embedding theorems, trace theorems
47A20	Dilations, extensions, compressions of linear operators
47A55	Perturbation theory of linear operators
47A60	Functional calculus for linear operators
47A63	Linear operator inequalities

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Besov classes; contractions; dissipative operators; double operator integrals; functions of noncommuting operators; operator Lipschitz estimates; Schatten classes; semi-spectral measures

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