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Norms on complex matrices induced by random vectors. (English) Zbl 07741833

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This paper introduces a family of norms on the space M_n of $n \times n$ complex matrices. These norms arise from a probabilistic framework as they are induced by random vectors whose entries are independent and identically distributed (iid) real-valued random variables with sufficiently many moments.

Initially, these norms are defined on complex Hermitian matrices as symmetric functions of their (necessarily real) eigenvalues. This contrasts with Schatten p-norms, which are defined in terms of singular values. To be more specific, these random vector norms do not arise from the machinery of symmetric gauge functions. Rather, they are generalizations of the complete homogeneous symmetric (CHS) polynomial norms introduced in [*K. Aguilar* et al., Bull. Lond. Math. Soc. 54, No. 6, 2078–2100 (2022; Zbl 07740261)].

The paper is organized as follows:

In Section 1, the preliminary concepts and notation are covered. Following this, the main result, which is lengthy and highly technical in nature, is stated.

Norms arising from familiar distributions (namely Gamma random variables, Normal random variables, Uniform random variables, Laplace random variables, Bernoulli random variables, Finite discrete random variables, Poisson random variables, and Pareto random variables) are examined in Section 2. Various examples and applications are also provided, including a powerful generalization of Hunter's positivity theorem for the complete homogeneous symmetric polynomials.

The proof of the main result, which involves a wide range of topics, such as cumulants, Bell polynomials, partitions, and Schur convexity, is contained in Section 3.

The paper concludes with a list of open questions.

Reviewer: Frédéric Morneau-Guérin (Québec)

MSC:

47A30	Norms (inequalities, more than one norm, etc.) of linear operators
15A60	Norms of matrices, numerical range, applications of functional analysis to matrix theory
16R30	Trace rings and invariant theory (associative rings and algebras)

Keywords:
norm; symmetric polynomial; partition; trace; positivity; convexity; expectation; complexification; trace polynomial; probability distribution

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