The diameter of the Birkhoff polytope

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Introduction

A square matrix $A = [a_{ij}]$ with nonnegative real entries is said to be *row-stochastic* (or *right stochastic*) if

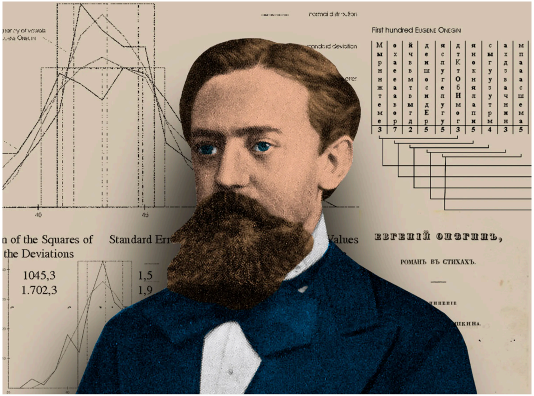
$$\sum_{j=1}^{n} d_{ij} = 1, \qquad \forall i = 1, \ldots, n.$$

Similarly, it is said to be *column-stochastic* (or *left stochastic*) if

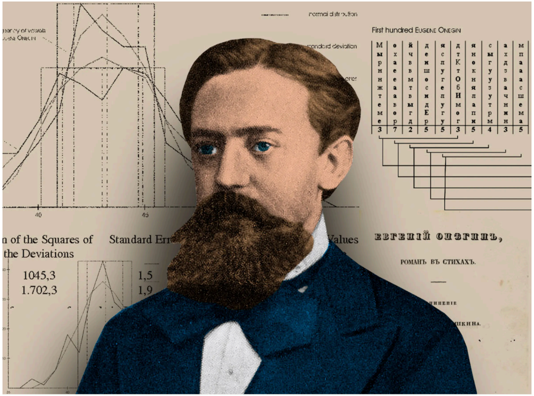
$$\sum_{i=1}^{n} d_{ij} = 1, \qquad \forall j = 1, \ldots, n.$$

A *doubly stochastic matrix* is one that is both row-stochastic and column-stochastic.

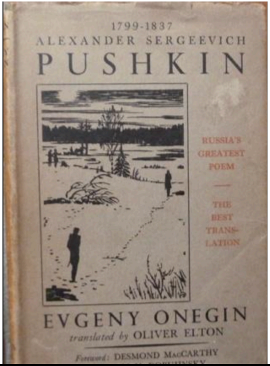
The theory of stochastic and doubly stochastic matrices was first developed alongside the Markov chain by the Russian mathematician Andrei Andreevich Markov (1856-1922) at the beginning of the 20th century.



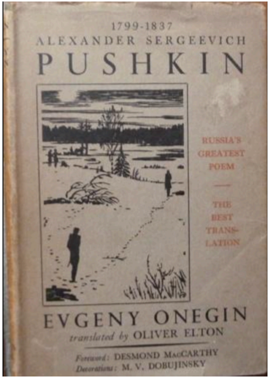
Markov first began developing his ideas about chains of linked events (where what happens next depends on the current state of the system) in 1906. The initial intended uses of the new branch of probability theory that he was elaborating were for linguistic analysis.



Delving into the text of Alexander Pushkin's novel in verse *Eugene Onegin*, Markov treated the text as a mere stream of letters. He spent hours sifting through patterns of vowels and consonants with the aim of estimating to what extent there is an exaggerated tendency in Pushkin's text for vowels and consonants to alternate, thus violating the principle of independence.



Although Markov’s analysis remains on a superficial level from a linguistic point of view, it made a lasting impression because the technique involved presciently extended the theory of probability in a new direction.



The ubiquity of stochastic and doubly stochastic matrices in science as a tool for statistical analysis alone justifies the study of these sets of matrices. But they are not without intrinsic interest.

1. Ω_n, the set of doubly stochastic matrices of size $n \times n$, forms a semigroup with respect to matrix multiplication;
2. Ω_n is a convex polytope (i.e., a compact convex set with a finite number of extreme points) in the Euclidean space of dimension n^2;
3. It was shown by G. Birkhoff (1946) that the extreme points of Ω_n are precisely the $n \times n$ permutation matrices, denoted by $\mathcal{P}_n$, and that each $D \in \Omega_n$ admits a (not necessarily unique) *Birkhoff decomposition* $D = \sum_{i=1}^{r} \alpha_i P_i$, where $P_i \in \mathcal{P}_n$, $\alpha_i \geq 0$, and $\sum_{i=1}^{r} \alpha_i = 1$. Due to this fundamental characterization, Ω_n is sometimes referred to as the *Birkhoff polytope*.

In the last few decades, the geometric features of the Birkhoff polytope have been an active subject of research.

The work described in the present talk is fully in line with that described by Ludovick in that the question addressed is centered around studying the geometry of the Birkhoff polytope when the ambient space is endowed with the metric induced by the operator norm from ℓ_n^p to ℓ_n^p $(1 \leq p \leq \infty)$ and, in turn, with the Schatten p-norm $(p \geq 1)$.

Definitions, Properties and Preliminary results

For $p \geq 1$, the *p-norm* of a given vector $x = (x_1, \ldots, x_n)$ is defined by

$$\|x\|_p = \left(\sum_{k=1}^{n} |x_k|^p \right)^{1/p},$$

and the ∞-norm is defined by

$$\|x\|_\infty = \max\{|x_1|, \ldots, |x_n|\}.$$

Any $n \times n$ matrix A can be interpreted as an operator from ℓ_n^p to ℓ_n^p. The *operator norm of A induced by the vector p-norm* is given by

$$\|A\|_{\ell_n^p \to \ell_n^p} := \sup_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p}.$$

For all $1 \leq p \leq \infty$, we have $\|B\|_{\ell_n^p \to \ell_n^p} = \|B^*\|_{\ell_n^q \to \ell_n^q}$, where B^* denotes the conjugate transpose of B and q is the Hölder conjugate exponent of p, i.e., $\frac{1}{p} + \frac{1}{q} = 1$.

It is also worth mentioning two properties of the operator norms induced by the vector p-norms which will prove useful in the remainder of this paper. Both easily derive from the definition. These are:

1. *Sub-multiplicativity*: For all $p \in [1, \infty]$ and all $n \times n$ matrices A and B,

$$\|AB\|_{\ell_n^p \to \ell_n^p} \leq \|A\|_{\ell_n^p \to \ell_n^p} \|B\|_{\ell_n^p \to \ell_n^p};$$

2. *Permutation invariance*: For all $p \in [1, \infty]$, all $n \times n$ matrix A, and all $n \times n$ permutation matrices P and Q,

$$\|PAQ\|_{\ell_n^p \to \ell_n^p} = \|A\|_{\ell_n^p \to \ell_n^p}.$$

Given an $n \times n$ matrix A, let $\lambda_1, \ldots, \lambda_n$ denote the eigenvalues of A^*A, with repetitions counted. Note that for all $i = 1, 2, \ldots, n$ we have

$$\lambda_i \|x\|_2^2 \; = \; \langle \lambda_i x, x \rangle \; = \; \langle A^* A x, x \rangle \; = \; \langle Ax, Ax \rangle \; \geq \; 0.$$

Hence, the λ_is are non-negative real numbers. Order these so that $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$. Let $\sigma_i := \sqrt{\lambda_i}$, so that $\sigma_1 \geq \cdots \geq \sigma_n \geq 0$. These latter numbers are called the *singular values* of A.

It follows from the Courant–Fischer min-max theorem that the i-th singular value of A is given by

$$\sigma_i(A) = \inf_{\substack{V \subseteq \mathbb{C}^n \\ \dim(V) = n-i+1}} \sup_{\substack{x \in V \\ \|x\|_2 = 1}} \|Ax\|_2, \qquad (i = 1, 2, \ldots, n).$$

It is easy to see that the largest singular value of A is equal to the operator norm of A induced by the vector 2-norm. More explicitly,

$$\sigma_1(A) = \|A\|_{\ell_n^2 \to \ell_n^2}.$$

For a given $p \in [1, \infty]$, the Schatten p-norm of an $n \times n$ matrix A, which we denote by $\|A\|$, is defined as the p-norm of the sequence of its singular values, that is

$$\|A\|_{\mathcal{S}_p} := \left(\sum_{i=1}^{n} \left|\sigma_i(A)\right|^p \right)^{1/p}, \qquad (1 \le p < \infty)$$

and

$$\|A\|_{\mathcal{S}_\infty} := \max\left\{ \left|\sigma_1(A)\right|, \ldots, \left|\sigma_n(A)\right| \right\} = \sigma_1(A).$$

Remark that the definition of the Schatten ∞-norm coincide with that of the *spectral norm* **??**.

Being closely related to vector p-norms, Schatten p-norms naturally inherit the following monotonic behaviour of the latter:

$$\|A\|_{\mathcal{S}_1} \;\geq\; \|A\|_{\mathcal{S}_p} \;\geq\; \|A\|_{\mathcal{S}_q} \;\geq\; \|A\|_{\mathcal{S}_\infty}$$

for all $n \times n$ matrix A and for $1 \leq p \leq q \leq \infty$.

Finally, the following two properties of Schatten p-norms are immediate consequences of the singular value decomposition (SVD). These are:

1. *Sub-multiplicativity*: For all $p \in [1, \infty]$ and all $n \times n$ matrices A and B,

$$\|AB\|_{\mathcal{S}_p} \leq \|A\|_{\mathcal{S}_p} \|B\|_{\mathcal{S}_p};$$

2. *Unitarily invariance (and a fortiori permutation invariance)*: For all $p \in [1, \infty]$, all $n \times n$ matrix A, and all $n \times n$ unitary matrices U and V,

$$\|UAV\|_{\mathcal{S}_p} = \|A\|_{\mathcal{S}_p}.$$

Elementary spectral properties of doubly stochastic matrices. Let us recall some basic yet useful facts concerning doubly stochastic matrices which will be needed later.

1. The spectrum of a doubly stochastic matrix always include 1. This eigenvalue is associated to the *obvious* eigenvector $e = (1, 1, \dots, 1)^\intercal$. All other eigenvalues have magnitude at most 1.
2. The spectrum of a doubly stochastic matrix D is contained within the unit circle if and only if D is a permutation matrix.
3. A convex real-valued function on Ω_n attains its maximum at a permutation matrix.

A special doubly stochastic matrix. The $n \times n$ matrix where every entry is equal to $1/n$, which we denote by J_n, plays a central role in the theory of doubly stochastic matrices and is quite special in a number of regards. But for the purposes of the following discussion it suffices to note that it acts as *the absorbing element* of Ω_n, i.e.,

$$DJ_n = J_n D = J_n$$

for every $n \times n$ doubly stochastic matrix D.

For the purposes of our study of the diameter of the Birkhoff polytope with respect to various norms, it will be useful to have upper and lower bounds for the value of these norms when the operand runs through the Birkhoff polytope. The following lemma provides such elementary bounds.

Lemma

Let $\|\cdot\|$ be a permutation invariant sub-multiplicative matrix norm and let D be an $n \times n$ doubly stochastic matrix. Then $1 \leq \|D\| \leq \|I_n\|$, where I_n is the $n \times n$ identity matrix.

Proof.

On the one hand, it follows from the absorbing property of the special matrix J_n that

$$\|J_n\| = \|DJ_n\| \leq \|D\|\|J_n\|. \tag{1}$$

Thus $\|D\| \geq 1$.

On the other hand, if $\sum_{i=1}^{r} \alpha_i P_i$ is a Birkhoff decomposition of D, then

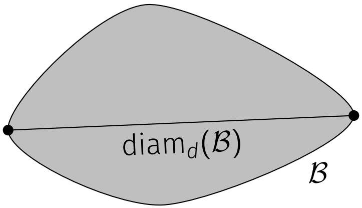
$$\|D\| = \left\| \sum_{i=1}^{r} \alpha_i P_i \right\| \leq \sum_{i=1}^{r} \alpha_i \|P_i\| = \sum_{i=1}^{r} \alpha_i \|I_n\| = \|I_n\|. \qquad \square$$

Definition of the diameter of a set

In any given metric space (X, d), the diameter of a nonempty set of points $B \subseteq X$ is defined as the supremum of the distances between pairs of points in B, i.e.,

$$\operatorname{diam}_d(\mathcal{B}) := \sup_{x,y \in \mathcal{B}} d(x, y).$$

If $\operatorname{diam}_d(\mathcal{B}) < \infty$, then A is called a bounded set.



The diameter of the nonempty closed bounded set $\mathcal{B}$ with respect to the metric d.

Diameter of the Birkhoff polytope relative to the operator norms from ℓ_n^p to ℓ_n^p for $1 \leq p \leq \infty$

Theorem

For every $1 \leq p \leq \infty$, $diam_{\ell_n^p}(\Omega_n) = 2$.

Proof.

One can easily check straight from the definition of the operator norm from ℓ_n^p to ℓ_n^p that $\|I_n\|_{\ell_n^p \to \ell_n^p} = 1$ for all $1 \leq p \leq \infty$. The Lemma therefore ensures that for any $D \in \Omega_n$, $\|D\|_{\ell_n^p \to \ell_n^p} = 1$. Thus,

$$\|D - S\|_{\ell_n^p \to \ell_n^p} \leq \|D\|_{\ell_n^p \to \ell_n^p} + \|S\|_{\ell_n^p \to \ell_n^p} = 2$$

for every $D, S \in \Omega_n$. Hence, $\mathrm{diam}_{\ell_n^p}(\Omega_n) \leq 2$. Moreover, let $D = I_{n-2} \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $S = I_n$. Then $D - S = 0_{n-2} \oplus \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$. The singular values of this matrix are 2 with multiplicity 1, and 0 with multiplicity $n - 1$. Consequently, the diameter of Ω_n relative to the operator norm from ℓ_n^p to ℓ_n^p is equal to 2. □

Remark: In the space of $n \times n$ matrices with real coefficients equipped with the metric induced by the $\|\cdot\|_{\ell_n^p \to \ell_n^p}$ norm, the Birkhoff polytope Ω_n has certain properties reminiscent of those of the unit circle. For example, $\|D\|_{\ell_n^p \to \ell_n^p} = 1$ for all $D \in \Omega_n$ and $\mathrm{diam}_p(\Omega_n) = 2$.

Note, however, that the unit circle also verifies a property that is easily taken for granted: its diameter is equal to twice its radius (in Chebyshev's sense). A set in a metric space that verifies both of the above properties of the unit circle as well as this latter property is called a *centrable* set.

It follows from Ludovick's talk that the Chebyshev radius of the Birkhoff polytope with respect to the metric induced by the spectral norm is 1, which makes Ω_n a centrable set with respect to this metric space. However, for $p \neq 2$, Ω_n is not a centrable set with respect to $\|\cdot\|_{\ell_n^p \to \ell_n^p}$ since its radius is strictly greater than 1.

Diameter of the Birkhoff polytope relative to the Schatten p-norm for $p \geq 1$

We begin by using the fact that $\|D-S\|_{\ell_n^p \to \ell_n^p}$ is convex relative to both operand $D, S \in \Omega_n$ to deduce that $\mathrm{diam}_{\mathcal{S}_p}(\Omega_n) = \mathrm{diam}_{\mathcal{S}_p}(\mathcal{P}_n)$. But, since Schatten p-norms are unitarily-invariant (and thus permutation-invariant), we have

$$\mathrm{diam}_{\mathcal{S}_p}(\Omega_n) = \max_{P,Q \in \mathcal{P}_n} \|P-Q\|_{\mathcal{S}_p} = \max_{P,Q \in \mathcal{P}_n} \|I_n - QP^*\|_{\mathcal{S}_p} = \max_{P \in \mathcal{P}_n} \|I_n - P\|_{\mathcal{S}_p}.$$

Now observe that $(I_n - P)^*(I_n - P) = 2I_n - P - P^*$ and that, if (λ, x) is an eigenpair of the permutation matrix P, then

$$(2I_n - P - P^*)x \;=\; (2 - \lambda - \lambda^{-1})x.$$

Thus, if $\lambda_k \ (1 \le k \le n)$ are the eigenvalues of P, then the singular values of $I_n - P$ are equal to $\sqrt{2 - \lambda_k - \lambda_k^{-1}}$. But, the eigenvalues of the permutation matrices are unimodular. So if $\lambda_k := e^{i\theta_k}$, we find

$$\sqrt{2 - \lambda_k - \lambda_k^{-1}} \;=\; \sqrt{2(1 - \cos(\theta_k))} \;=\; 2\sin\left(\tfrac{\theta_k}{2}\right), \qquad (0 \le \theta_k \le 2\pi).$$

It follows that

$$\|I_n - P\|_{\mathcal{S}_p} \;=\; 2\left(\sum_{k=1}^{n} \sin^p\left(\tfrac{\theta_k}{2}\right)\right)^{\frac{1}{p}},$$

where the θ_k are the arguments of the eigenvalues of P.

Recall that the eigenvalues of a permutation matrix are of following particular form: there exist natural numbers $n_1, n_2, \ldots, n_r$ satisfying $n_1 + n_2 + \cdots + n_r = n$ for which the eigenvalues of P are $e^{2\pi i k/n_1}$ $(1 \le k \le n_1)$, $e^{2\pi i k/n_2}$ $(1 \le k \le n_2)$, $\ldots$, $e^{2\pi i k/n_r}$ $(1 \le k \le n_r)$. Therefore,

$$\|I_n - P\|_{\mathcal{S}_p} = 2\left(\sum_{k=1}^{n_1} \sin^p\left(\frac{\pi k}{n_1}\right) + \sum_{k=1}^{n_2} \sin^p\left(\frac{\pi k}{n_2}\right) + \cdots + \sum_{k=1}^{n_r} \sin^p\left(\frac{\pi k}{n_r}\right)\right)^{\frac{1}{p}}$$

and thus

$$\operatorname{diam}_{\mathcal{S}_p}(\Omega_n) = 2 \max_{\sum n_j = n} \left(\sum_{k=1}^{n_1} \sin^p\left(\frac{\pi k}{n_1}\right) + \cdots + \sum_{k=1}^{n_r} \sin^p\left(\frac{\pi k}{n_r}\right)\right)^{\frac{1}{p}}.$$

Finding this maximum is a difficult problem. Indeed, a partition maximizing the above quantity not only depends on the parameter p, but also, as we shall see, on the parity of n when $p > 2$. We therefore divide the problem into three subcases. In the first case, where $1 \leq p < 2$, we will use some non-trivial identities to achieve our goal. In the second case, certain peculiarities of the Schatten 2-norm will be exploited to find precisely for which doubly stochastic matrices the diameter is realized. Finally, we will characterize the case $p > 2$ using estimates.

Subcase I : $1 \le p < 2$. For every $p \in [0,2)$, the function

$$S(n) := \frac{1}{n}\sum_{k=1}^{n} \sin^p\left(\frac{\pi k}{n}\right)$$

is monotonically increasing relative to the natural numbers n. It thus follows that

$$\begin{aligned}
\operatorname{diam}_{\mathcal{S}_p}(\Omega_n) &= 2 \max_{\sum n_j = n} \left(\sum_{k=1}^{n_1} \sin^p\left(\tfrac{\pi k}{n_1}\right) + \cdots + \sum_{k=1}^{n_r} \sin^p\left(\tfrac{\pi k}{n_r}\right) \right)^{\frac{1}{p}} \\
&= 2 \max_{\sum n_j = n} \left(n_1 S(n_1) + \cdots + n_r S(n_r)\right)^{\frac{1}{p}} \\
&\le 2 \max_{\sum n_j = n} \left(n_1 S(n) + \cdots + n_r S(n)\right)^{\frac{1}{p}} \\
&= 2\left(nS(n)\right)^{\frac{1}{p}} = 2\left(\sum_{k=1}^{n} \sin^p\left(\tfrac{\pi k}{n}\right) \right)^{\frac{1}{p}}.
\end{aligned}$$

The reverse inequality being realized by the trivial partition (n), it follows that

$$\operatorname{diam}_{\mathcal{S}_p}(\Omega_n) = 2\left(\sum_{k=1}^{n} \sin^p\left(\frac{\pi k}{n}\right)\right)^{\frac{1}{p}}, \qquad (1 \leq p < 2).$$

Subcase II : $p = 2$.

Theorem

The diameter of Ω_n relative to the Schatten 2-norm is equal to $\sqrt{2n}$.

Proof.

Let $D_1, D_2 \in \Omega_n$. We have

$$\begin{aligned}\|D_1 - D_2\|_{\mathcal{S}_2}^2 &= \|D_1\|_{\mathcal{S}_2}^2 + \|D_2\|_{\mathcal{S}_2}^2 - 2\mathrm{tr}(D_1 D_2^*)\\ &\leq 2\|I_n\|_{\mathcal{S}_2}^2 - 2\mathrm{tr}(D_1 D_2^*)\\ &\leq 2n.\end{aligned}$$

Taking the supremum on each side yield $\mathrm{diam}_{\mathcal{S}_2}(\Omega_n) \leq \sqrt{2n}$. We easily obtain the reverse inequality by taking the pair of doubly stochastic matrices I_n and Q, where Q is any permutation matrix with trace zero. Hence, $\mathrm{diam}_{\mathcal{S}_2}(\Omega_n) = \sqrt{2n}$. $\square$

Subcase III : $p > 2$.

Clearly, $\sin^p(x) \leq \sin^2(x)$ for any $x \in [0, \pi]$ and $p > 2$. Moreover, it is easy to show (using Euler identity) that $\sum_{k=1}^{n} \sin^2(\frac{\pi k}{n}) = \frac{n}{2}$ for $n \geq 2$ and $\sum_{k=1}^{1} \sin^2(\frac{\pi k}{1}) = 0$. The combination of these basic facts gives us the following estimate:

$$\begin{aligned} \sum_{k=1}^{n_1} \sin^p(\tfrac{\pi k}{n_1}) + \cdots + \sum_{k=1}^{n_r} \sin^p(\tfrac{\pi k}{n_r}) &\leq \sum_{k=1}^{n_1} \sin^2(\tfrac{\pi k}{n_1}) + \cdots + \sum_{k=1}^{n_r} \sin^2(\tfrac{\pi k}{n_r}) \\ &\leq \frac{n_1}{2} + \cdots + \frac{n_r}{2} = \frac{n}{2}. \end{aligned}$$

It follows that $\operatorname{diam}_{\mathcal{S}_p}(\Omega_n) \leq 2\left(\frac{n}{2}\right)^{\frac{1}{p}}$ for $p > 2$. If n is even, the partition of n into 2s yields the reverse inequality and thus,

$$\operatorname{diam}_{\mathcal{S}_p}(\Omega_n) = 2\left(\frac{n}{2}\right)^{\frac{1}{p}}, \qquad (p > 2,\ n \text{ even}).$$

If n is odd, the situation is trickier. Clearly any partition of n must contain at least one odd term, say $n_r = m$. Proceeding as before, we have

$$\begin{aligned}
\operatorname{diam}_{\mathcal{S}_p}(\Omega_n) &= 2 \max_{\sum n_j = n} \left(\sum_{k=1}^{n_1} \sin^p\left(\frac{\pi k}{n_1}\right) + \cdots + \sum_{k=1}^{n_{r-1}} \sin^p\left(\frac{\pi k}{n_{r-1}}\right) + \sum_{k=1}^{m} \sin^p\left(\frac{\pi k}{m}\right) \right)^{\frac{1}{p}} \\
&\leq 2 \max_{\sum n_j = n} \left(\frac{n_1}{2} + \cdots + \frac{n_{r-1}}{2} + \sum_{k=1}^{m} \sin^p\left(\frac{\pi k}{m}\right) \right)^{\frac{1}{p}} \\
&= 2 \max_{\substack{1 \leq m \leq n \\ m \text{ odd}}} \left(\frac{n-m}{2} + \sum_{k=1}^{m} \sin^p\left(\frac{\pi k}{m}\right) \right)^{\frac{1}{p}}.
\end{aligned}$$

The reverse inequality is readily obtained by considering the partition $(2, \ldots, 2, m)$, and thus

$$\operatorname{diam}_{\mathcal{S}_p}(\Omega_n) = 2 \max_{\substack{1 \leq m \leq n \\ m \text{ odd}}} \left(\frac{n-m}{2} + \sum_{k=1}^{m} \sin^p\left(\tfrac{\pi k}{m}\right) \right)^{\frac{1}{p}} = 2\left(\frac{n}{2} + \max_{\substack{1 \leq m \leq n \\ m \text{ odd}}} S_p(m) \right)^{\frac{1}{p}},$$

where

$$S_p(m) := \sum_{k=1}^{m} \sin^p\left(\frac{\pi k}{m}\right) - \frac{m}{2}.$$

We therefore seek to determine for which odd numbers m the term $S_p(m)$ (seen as a function of p) attains its maximum.

A few clever trigonometric sleight of hands later...

Theorem

For $p > 2$, the diameter of Ω_n relative to the Schatten p-norm is

$$diam_{\mathcal{S}_p}(\Omega_n) = 2\left(\frac{n - \sin^2(\frac{n\pi}{2})\min\left\{1, 3 - 4\left(\sqrt{3}/2\right)^p\right\}}{2}\right)^{\frac{1}{p}}. \tag{2}$$

In particular, if n is even, then $diam_{\mathcal{S}_p}(\Omega_n) = 2(n/2)^{1/p}$.

To sum up

Theorem

Given p with $p \geq 1$, the diameter of Ω_n relative to the Schatten p-norm is given by

$$\mathrm{diam}^p_{\mathcal{S}_p}(\Omega_n) = \begin{cases} 2^p \sum_{k=1}^{n} \sin^p\left(\frac{\pi k}{n}\right), & \text{if } 1 \leq p \leq 2, \\ 2^p \frac{n - \sin^2\left(\frac{n\pi}{2}\right) \min\left(1, 3 - 4\left(\sqrt{3}/2\right)^p\right)}{2}, & \text{if } 2 \leq p < \infty. \end{cases} \quad (3)$$

In particular,

1. $\mathrm{diam}_{\mathcal{S}_1}(\Omega_n) = 2\cot\left(\frac{\pi}{2n}\right)$,
2. $\mathrm{diam}_{\mathcal{S}_2}(\Omega_n) = \sqrt{2n}$,
3. $\mathrm{diam}_{\mathcal{S}_p}(\Omega_n) = 2(n/2)^{1/p}$ *if n is even and $p \geq 2$.*

References

1. Bouthat, L., Mashreghi, J., & Morneau-Guérin, F. (2022). Monotonicity of certain left and right Riemann sums. In Recent Developments in Operator Theory, Mathematical Physics and Complex Analysis: *IWOTA 2021, Chapman University* (pp. 89-113). Cham: Springer International Publishing.
2. Bouthat, L., Mashreghi, J., & Morneau-Guérin, F. (2023). On the norm of normal matrices. *RIMS Kôkyûroku Bessatsu*, *93*, 183-222.
3. Bouthat, L., Mashreghi, J., & Morneau-Guérin, F. (submitted). On the Geometry of the Birkhoff Polytope. I. The operator ℓ_n^p-norms. *arXiv preprint arXiv:2310.14041.*
4. Bouthat, L., Mashreghi, J., & Morneau-Guérin, F. (submitted). On the Geometry of the Birkhoff Polytope. II. The Schatten p-norms. *arXiv preprint arXiv:2310.14043.*
5. Bouthat, L., Mashreghi, J., & Morneau-Guérin, F. (submitted). The Diameter of the Birkhoff Polytope. *arXiv preprint arXiv:2310.14043.*

Thank you!

Any Questions?