# Exact Short Products From Truncated Multipliers 

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#### Abstract

We sometimes need to compute the most significant digits of the product of small integers with a multiplier requiring much storage: e.g., a large integer (e.g., $5^{100}$ ) or an irrational number ( $\pi$ ). We only need to access the most significant digits of the multiplier-as long as the integers are sufficiently small. We provide an efficient algorithm to compute the range of integers given a truncated multiplier and a desired number of digits.


Keywords: Modular Arithmetic, Truncated Multiplication, Short Product

## 1. INTRODUCTION

In applications such as cryptography [1], digital signal processing [2], number serialization [3, 4] or number parsing [5], we need to efficiently compute the most significant digits or bits (binary digits) of a product. We call such a partial product a short product.

We consider the computation of the product between small integers and a multiplier with many digits. To illustrate the problem, suppose that we want to compute the product of $\pi$ with an integer and get 10digit accurate answer. With the 13 most significant digits of $\pi$ (3.141592653589), we get an accurate product for all integers in $[1,1198)$ no matter the missing digits of $\pi$. See Table 1

We are given only the most significant digits of the multiplier (i.e., a short multiplier) and we want to know whether we can compute the most significant digits of the product exactly, for some range of values of $w$. We provide an efficient logarithmic-time algorithm to compute the exact range of validity (§ 6.1). We make available our algorithms as part of an open-source Python library ${ }^{11}$

## 2. RELATED WORK

Given only a short multiplier of $z$, we seek the exact range of values $w$ such that we can compute a given number of most significant digits of the product $w \times z$ using the short multiplier. One potential application of our work is in number parsing: given the string 3 e 100 , we may want to convert it to a 64 -bit binary floating-point number: $3 \times 10^{100} \approx 7721336384202043 \times$

[^0]TABLE 1: Range of integers $w$ such that the 10 most significant digits of $w \times \pi$ are exact as a function of the number of most significant digits of $\pi$ used. The bound is blind to the missing digits of $\pi$.

| digits of $\pi$ | interval for 10-digit accuracy |
| :---: | :--- |
| 10 | $[1,2)$ |
| 11 | $[1,14)$ |
| 12 | $[1,209)$ |
| 13 | $[1,1198)$ |
| 14 | $[1,18149)$ |
| 15 | $[1,26255)$ |
| 16 | $[1,1454833)$ |
| 17 | $[1,14920539)$ |
| 18 | $[1,14920539)$ |
| 19 | $[1,1963319607)$ |
| 20 | $[1,17329613732)$ |

$2^{281}$ where 7721336384202043 is the 53 -bit significand chosen to best approximate $3 \times 10^{100}$. We have that $3 \times 10^{100}=3 \times 5^{100} \times 2^{100}$. Ignoring the power of two, the significand of the binary floating-point number (7721336384202043) may be computed by multiplying the decimal significand (3) by the power of five ( $5^{100}$ ) and selecting the 53 most significant bits. For speed, we may want to avoid computing $5^{100}$ and the full product $3 \times 5^{100}$. Instead, we want to just store the most significant bits of $5^{100}$ 5. However, we also need to check the validity of the short multipliers to ensure that the computation of the most significant digits is exact.

To our knowledge, our problem, the exact computation of the range of validity of a short multiplier, is
novel. Adams [3, 4] considered a related problem in the context of number serialization. They bound the maximum and minimum of $a x \% b$ over an interval starting at zero. With these bounds, they show that powers of five truncated to 128 bits are sufficient to convert 64bit binary floating-point numbers into equivalent decimal numbers. Specifically, Lemma 3.6 from Adams 3 computes a conservative approximation of the true minimum and maximum of $a x \% b$ for $x \in[0, M]$. In contrast, we present a logarithmic-time algorithm (in § 7) that provides all of the minima and maxima. This exact result allows us to compute an exact range of validity for a short multiplier.

## 3. MATHEMATICAL PRELIMINARIES

We present our core mathematical notation and we review some elementary results. See Table 2] For simplicity, we avoid references to equivalence classes or other extraneous concepts.

Let $\lfloor x\rfloor$ be the largest integer smaller or equal to $x$. For a number $a$ and another number $b \neq 0$, we define the integer division $a \div b \equiv\lfloor a / b\rfloor$ and the remainder $a \% b \equiv a-(a \div b) \times b$.

We say that $b \neq 0$ divides $a$ if $a \% b=0$. We write the greatest common divisor of integers $a$ and $b$ as $\operatorname{gcd}(a, b)$. We say that $a$ and $b$ are coprime if $\operatorname{gcd}(a, b)=1$.

The smallest integer $a^{\prime}$ such that $a \div b=a^{\prime} \div b$ is $a^{\prime}=(a \div b) \times b=a-(a \% b)$. We have that $(a+b) \% M=(a \% M+b \% M) \% M$.

Consider a positive integer divisor $M$. When $a \% M=0$ then $(-a) \% M=0$. Otherwise, when $a \% M \neq 0$, then we have $(-a) \% M=M-(a \% M)$.

The distance between two integers $a, b$ is often defined as the absolute value of their difference $|a-b|$. We use a generalized measure:

$$
\operatorname{distance}_{M}(a, b) \equiv \min ((a-b) \% M,(b-a) \% M)
$$

Because $(a+b) \% M=(a \% M+b \% M) \% M$, we have the following elementary results:

- $a \% M+b \% M<M$ if and only if $a \% M+b \% M=$ $(a+b) \% M$,
- $\quad a \% M+b \% M \geq M$ if and only if $a \% M+b \% M=$ $M+(a+b) \% M$.

Similarly, we have:

- $\quad a \% M+b \% M<M$ if and only if $(a+b) \% M \geq$ $\max (a \% M, b \% M)$,
- $\quad a \% M+b \% M \geq M$ if and only if $(a+b) \% M<$ $\min (a \% M, b \% M)$.

Further if $a \% M>0$ and $b \% M>0$ then $a \% M+b \% M<M$ if and only if $(a+b) \% M>$ $\max (a \% M, b \% M)$. Given two integers $a, b$ and an integer divisor $M$, we either have $(a-b) \% M=$ $a \% M-b \% M$ when $a \% M \geq b \% M$, or $(a-b) \% M=$ $M+a \% M-b \% M$ otherwise.

| symbol | meaning |
| :---: | :--- |
| $\lfloor x\rfloor$ | largest integer no larger than $x$ |
| $\operatorname{gcd}(a, b)$ | greatest common divisor of $a$ and $b$ |
| $a \div b$ | integer division of $a$ by $b:\lfloor a / b\rfloor$ |
| $a \% b$ | remainder of the division of $a$ by $b$ |
| $z$ | a multiplier (e.g., large integer) |
| $w$ | integer to be multiplied by $z$ |
| $\alpha$ | integer corresponding to a minimum |
| $\beta$ | integer corresponding to a maximum |
| $M$ | integer divisor |
| distance $_{M}(a, b)$ | $\min ((a-b) \% M,(b-a) \% M)$ |

TABLE 2: Notational conventions.

## 4. PLAN

Our derivation is organized as follow.

1. We formalize the concept of most significant digits of a product in § 5. Unsurprisingly, the computation of the most significant digits depends on the size of the product. Effectively, we get the most significant digits by dividing by some power of the base (e.g., $10^{3}$ ), and by discarding the remainder.
2. In § 6, we show that a short multiplier always provides an exact short product if and only if the discarded remainder is not too large (Lemma 6.1). These remainders take the form $(w \times z) \div M$ where $w$ is an integer variable while $z$ and $M$ are integer constants. By combining the result from the previous section (§5), we describe how checking for exact most-significant digits requires to bound various remainders over ranges. We conclude this section with a technical lemma (Lemma 6.2) which suggests that identifying the maxima of remainders is sufficient. Treating $w \rightarrow(w \times z) \div M$ as a function, we need to identify the values of $w$ that provide a new maximum for the expression $(w \times$ $z) \div M$ as $w$ is incremented (e.g., $w=1,2,3, \ldots$ ).
3. In $\S 7$ we present Lemma 7.1 which says that if we know the location of the last maximum $(w=\beta)$ and the last minimum $(w=\alpha)$, then the next extremum is at the sum of the two $(w=\alpha+\beta)$. This leads us to an efficient (logarithmic-time) algorithm to locate all extrema. The result is an algorithm that we present in Python (Fig. 21): it computes the gaps between the extrema of $(w \times$ z) $\% M$ for $w=1,2, \ldots$ In $\S 7.1$, we show that these gaps are sufficient to locate efficiently the extrema of $(w \times z) \% M$ over a range $(w=A, A+1, \ldots, B)$ that does not begin at $1(w=1)$. Finally, we provide a function find_max_min (Fig. 3) to enumerate all extrema of $(w \times z) \% M$ for $w=$ $A, \ldots, B$.
4. In § 8, we combine the function find_max_min with the results from § 6 to arrive at our main function (find_range) presented in Fig. 3. Given
a short multiplier, and a desired number of digits, it computes the range of validity.

## 5. MOST SIGNIFICANT DIGITS

We often represent integers with digits. E.g., the integer 1234 has four decimal digits. The integer 7 requires three binary digits. The number of digits of the integer $x$ in base $b$ is the smallest integer $d$ such that $x \div b^{d}=0$. By convention, the integer 0 has no digit (zero digit) and we do not consider negative integers. We may compute the number of digits in base $b$ of a positive integer $x$ using the formula $\left\lceil\log _{b}(x+1)\right\rceil$. In base 10 , the integers with three digits go from 100 to 999 , or from $10^{2}$ to $10^{3}-1$, inclusively. More generally, an integer has $d$ digits in base $b$ if it is between $b^{d-1}$ and $b^{d}-1$, inclusively.

The product between an integer having $d_{1}$ digits and integer having $d_{2}$ digits is between $b^{d_{1}+d_{2}-2}$ and $b^{d_{1}+d_{2}}-b^{d_{1}}-b^{d_{2}}+1$. (inclusively). Thus the product has either $d_{1}+d_{2}-1$ digits or $d_{1}+d_{2}$ digits. To illustrate, let us consider the product between two integers having three digits. In base 10 , the smallest product is 100 times 100 or 10000 , so it requires 5 digits. The largest product is 999 times 999 or 998001 ( 6 digits).

Suppose that we want $d \geq 1$ digits of accuracy (in base $b$ ) for the integer product $w \times z$ given a fixed integer $z$. As much as possible, we want to compute the $d$ most significant digits:

- If $b^{d-1} \leq(w \times z) \leq b^{d}-1$, then we output $w \times z$.
- If $b^{d} \leq(w \times z) \leq b^{d+1}-1$, then we output $(w \times z) \div b$.
- $\quad$ Generally, if $b^{d+k-1} \leq(w \times z) \leq b^{d+k}-1$ for $k \geq 0$, then we output $(w \times z) \div b^{k}$.


## 6. SHORT MULTIPLIERS

Suppose that we want to compute the most significant digits of $w \pi$ for small integers $w$. Materializing all digits of $\pi$ is impossible, so we use a truncated version: $3.1416 \times w$. Thus we may compute the most-significant digits of $3.1416 \times w$. For simplicity, we can omit the decimal point. How large could the integer $w$ be if we want two digits of accuracy assuming we do not use the missing digits of $\pi$ ? The answer is that $w$ should not exceed 2068. It is an instance of the general question we want to be able to answer.

When $10^{3} \leq 31416 \times w \leq 10^{4}-1,(31416 \times$ $w) \div 10$ provides the two-most significant digits of the product. We want to determine when $(31416 \times w) \div 10$ matches the value we would get when computing the full product: $(31416 \times w) \div 10=(10000 \pi \times w) \div 10$ irrespective of the truncated portion of $\pi(10000 \pi-$ 31416). The following lemma provides a necessary and sufficient condition.

Lemma 6.1. Let $z$ be a truncated integer multiplier and $z^{\prime}=z+\epsilon$ be the exact multiplier with $\epsilon \in[0,1)$.

For integers $w$ and $M$, we have that $(w \times z) \div M=(w \times$ $\left.z^{\prime}\right) \div M$ for all $\epsilon$ if and only if $(w \times z) \% M<M-w+1$.
Proof. Let $z$ be the truncated integer multiplier and $z^{\prime}=z+\epsilon$ be the exact multiplier with $\epsilon \in[0,1)$. Since $z \leq z^{\prime}<z+1$, we have that $w \times z \leq w \times z^{\prime}<w \times z+w$. Hence we have that $w \times z \leq w \times z^{\prime} \leq w \times z+w-1$. Thus we have $(w \times z) \div M \leq\left(w \times z^{\prime}\right) \div M \leq(w \times z+w-1) \div M$. Therefore, we have that $(w \times z) \div M=\left(w \times z^{\prime}\right) \div M$ for all $z^{\prime}$ if and only if $(w \times z) \div M=(w \times z+w-1) \div M$. Write $(w \times z+w-1) \div M=((w \times z) \div M \times M+(w \times$ z) $\% M+w-1) \div M$. With a little arithmetic, we see from this last equality that $(w \times z) \% M+w-1<M$ if and only if $(w \times z+w-1) \div M=(w \times z) \div M$. We have proven the lemma.

We can apply this condition to the computation of digits. Consider $w \times z$ and suppose you desire to have $d \geq 1$ digits of accuracy in base $b$.

- If $(w \times z) \leq b^{d-1}-1$, then we cannot produce $d$ digits by truncation. Thus we may require as a pre-condition that $(w \times z) \geq b^{d-1}$ or $w \geq$ $\left(b^{d-1}+z-1\right) \div z$.
- If $b^{d-1} \leq(w \times z) \leq b^{d}-1$, then we output $w \times z$ after checking that $w<2$.
- If $b^{d} \leq(w \times z) \leq b^{d+1}-1$, then we output $(w \times z) \div b$ after checking that $(w \times z) \% b<b-w+1$.
- ...
- Generally, if $b^{d+k-1} \leq(w \times z) \leq b^{d+k}-1$ for $k \geq 0$, then we output $(w \times z) \div b^{k}$ after checking that $(w \times z) \% b^{k}<b^{k}-w+1$.
We would like to check $(w \times z) \% b^{k}<b^{k}-w+1$ efficiently over the range $b^{d+k-1}-1<(w \times z) \leq b^{d+k}-1$ to verify whether we can compute digits exactly.


### 6.1. Finding the Valid Range

Suppose that $(\beta \times z) \% M$ is the maximum of $(w \times$ $z) \% M$ for $w=0,1, \ldots, \beta$, then it follows that if $(\beta \times z) \% M \leq M-\beta+1$ is satisfied, it must be that $(w \times z) \% M<M-w+1$ is satisfied for $w=$ $0,1, \ldots, \beta-1$, since $M-w+1$ is strictly decreasing and $(w \times z) \% M<(\beta \times z) \% M$. Conversely, the next lemma shows that the first time that $(w \times z) \% M<M-w+1$ is falsified, $(w \times z) \% M$ is a new maximum.

Lemma 6.2. Let $w^{\prime}$ be the smallest integer value $w^{\prime} \geq$ 0 such that $\left(w^{\prime} \times z\right) \% M \geq M-w^{\prime}+1$, then we have that $\left(w^{\prime} \times z\right) \% M>(w \times z) \% M$ for $w=0,1, \ldots, w^{\prime}-1$.
Proof. Let $w_{2}$ be the smallest value $w$ such that $(w \times$ z) $\% M<M-w+1$ is falsified. Let $w_{1}$ be the location of the maximum of $(w \times z) \% M$ up to $w_{2}$ exclusively: $0 \leq w_{1}<w_{2}$. Then we have $w_{1} \leq w_{2}$ such that $\left(w_{1} \times z\right) \% M \geq\left(w_{2} \times z\right) \% M,\left(w_{1} \times z\right) \% M<M-w_{1}+1$ and $\left(w_{2} \times z\right) \% M \geq M-w_{2}+1$. Thus we have that $\left(w_{1} \times z\right) \% M-\left(w_{2} \times z\right) \% M<w_{2}-w_{1}$ or $\left(w_{2} \times z\right) \% M-\left(w_{1} \times z\right) \% M>w_{1}-w_{2}$ or $M+\left(w_{2} \times\right.$ z) $\% M-\left(w_{1} \times z\right) \% M>M+w_{1}-w_{2}$. Thus we have

| $w$ | $(w \times 3) \% 8$ | classification |
| :---: | :---: | :---: |
| 1 | 3 | maximum/minimum |
| 2 | 6 | maximum |
| 3 | 1 | minimum |
| 4 | 4 |  |
| 5 | 7 | maximum |
| 6 | 2 |  |
| 7 | 5 | minimum |
| 8 | 0 |  |

TABLE 3: Example of remainders with $M=8$ and $z=3$.
that $\left(\left(w_{2}-w_{1}\right) \times z\right) \% M>M-\left(w_{2}-w_{1}\right)$. This indicates that $w_{2}-w_{1}$ falsifies $(w \times z) \% M<M-w+1$, which is only possible if $w_{1}=0$ by our assumption, but that is not possible since it would imply that the maximum is 0 . We have shown the lemma.

This lemma is helpful because it indicates that we only need to check the condition $(w \times z) \% M \geq M-$ $w+1$ when $(w \times z) \% M$ is a new maximum.

Given a short multiplier $z$ and a desired number of digits $d$ in base $b$, we may seek the upper range of the variable $w$ such that the $d$ most significant digits of $w \times z$ are exact. We iterate over $k=0,1, \ldots$ and for values of $w$ such that $b^{d+k-1}-1<(w \times z) \leq b^{d+k}-1$, we seek the smallest value $w$ such that $(w \times z) \% b^{k} \geq b^{k}-w+1$. When such a value exists, the algorithm terminates with a value $w$ indicating the upper bound. The lower bound is given by $w \geq\left(b^{d-1}+z-1\right) \div z$. The lower and upper bounds define the range of validity: given any integer $w$ in this range, the $d$ most significant digits of $w \times z$ are exact.

## 7. ENUMERATING THE EXTREMA OF REMAINDERS

As we compute $(w \times z) \% M$ for $w=1,2,3, \ldots, M-1$, we encounter new minima and new maxima. We seek to efficiently locate all such extrema.

Consider $M=8$ and $z=3$. Given the first two values, $(1 \times 3) \% 8=3$ and $(2 \times 3) \% 8=6$, we have that the former is a minimum while the later is a maximum. See Table 3. The next value at $w=3$ is 1 , a new minimum. The next extrema is at $w=5$, a maximum. Observe that prior to $w=5$, we had a maximum at $w=2$ and a minima at $w=3$ and that $3+2=5$. As we shall show, all extrema follow this rule: they appear at a location that is the sum of the location of last minimum with the last maximum.

When $z$ and $M$ are coprime, then the sequence of values $(w \times z) \% M$ for $w=1,2,3, \ldots, M-1$ are a permutation of the integers from 1 to $M-1$. When $z$ and $M$ have a non-trivial common divisor, the values repeat. Indeed, whenever $(w \times z) \% M=\left(w^{\prime} \times z\right) \% M$, we have that $\left(\left(w-w^{\prime}\right) \times z\right) \% M=0$. Assume without loss of generality that $w>w^{\prime}$, then we have
that $\left(w-w^{\prime}\right) \times z$ is a positive integer divisible by $M$. Thus we have that the sequence $(w \times z) \% M$ for $w \in[0, M / \operatorname{gcd}(M, z))$ is made of distinct values. This sequence of values repeats over the next interval $[M / \operatorname{gcd}(M, z), 2 M / \operatorname{gcd}(M, z))$ and so forth, $\operatorname{gcd}(M, z)$ times until $w=M$.

The next lemma shows how we can always determine the location of the next extrema from the last minimum and the last maximum. If the last minimum is at $w=\alpha$ and the last maximum is at $w=\beta$, then the next extrema is at $w=\alpha+\beta$. See Fig. (1).

Lemma 7.1. Suppose that, over a range $w=$ $1,2, \ldots, \max (\beta, \alpha)$ for $\max (\beta, \alpha)<M / \operatorname{gcd}(M, z)$, we have that $(\beta \times z) \% M$ is the maximal value of $(w \times$ $z) \% M$, while $(\alpha \times z) \% M$ is the minimal value, then if we extend the sequence to $w=1,2, \ldots, \alpha+\beta$, we have that

- When $((\alpha+\beta) \times z) \% M>(\beta \times z) \% M,((\alpha+$ $\beta) \times z) \% M$ is the new maximum while $(\alpha \times z) \% M$ remains the minimum,
- otherwise we have that $((\alpha+\beta) \times z) \% M<$ $(\beta \times z) \% M$, and $((\alpha+\beta) \times z) \% M$ is the new minimum while $(\beta \times z) \% M$ remains the maximum.

Proof. For brevity, assume that $\alpha<\beta$ : the counterpart $(\alpha>\beta)$ follows by symmetrical arguments.

We want to show that $((\alpha+\beta) \times z)$ is either smaller than $(\alpha \times z) \% M$ or larger than $(\beta \times z) \% M$ :

- Suppose that $((\alpha+\beta) \times z) \% M \geq(\alpha \times z) \% M$ then $((\alpha+\beta) \times z) \% M=(\alpha \times z) \% M+(\beta \times z) \% M>$ $(\beta \times z) \% M$.
- Suppose that $((\alpha+\beta) \times z) \% M \leq(\beta \times z) \% M$ then $((\alpha+\beta) \times z) \% M=(\alpha \times z) \% M+(\beta \times z) \% M-$ $M<(\alpha \times z)$.

First suppose that $((\alpha+\beta) \times z) \% M>(\beta \times z) \% M$. We want to show that $((\alpha+\beta) \times z) \% M$ is the new maximum over the extended range while $(\alpha \times z) \% M$ remains the minimum. Suppose that it is not the case, then at least one of the following two cases hold:

- If $((\alpha+\beta) \times z) \% M$ is not the new maximum, there must be a new, even larger maximum. Thus there must be a value $\omega$ satisfying $0<\omega<\alpha$ such that $((\omega+\beta) \times z) \% M>((\alpha+\beta) \times z) \% M>$ $(\beta \times z) \% M$. It implies that $((\omega+\beta) \times z) \% M=$ $((\omega-\alpha) \times z) \% M+((\alpha+\beta) \times z) \% M=((\omega-\alpha) \times$ $z) \% M+(\alpha \times z) \% M+(\beta \times z) \% M=((\omega+\beta-\alpha) \times$ $z) \% M+(\alpha \times z) \% M$. Hence we have that $((\omega+\beta-$ $\alpha) \times z) \% M=((\omega+\beta) \times z) \% M-(\alpha \times z) \% M>$ $((\alpha+\beta) \times z) \% M-(\alpha \times z) \% M=(\beta \times z) \% M$. Hence we have that $((\omega+\beta-\alpha) \times z) \% M>$ $(\beta \times z) \% M$ which contradicts that $\beta$ is a maximum for the range up to $\beta$ since $\omega+\beta-\alpha<\beta$. Thus this case is not possible.
- If $(\alpha \times z) \% M$ did not remain the minimum, then there must be a new, even smaller minimum: there


FIGURE 1: Illustration of how the next extrema is computed from the last minimum and the last maximum
must be a value $\omega$ satisfying $0<\omega<\alpha$ such that $((\omega+\beta) \times z) \% M<(\alpha \times z) \% M<(\beta \times z) \% M$. It implies that $(\omega \times z) \% M+(\beta \times z) \% M-M<$ $(\alpha \times z) \% M$ or $(\alpha \times z) \% M-(\omega \times z) \% M>$ $(\beta \times z) \% M-M$. Because $0<\omega<\alpha$, we must have that $(\omega \times z) \% M>(\alpha \times z) \% M$. Thus we have that $((\alpha-\omega) \times z) \% M$ is $M+(\alpha \times z) \% M-$ $(\omega \times z) \% M>(\beta \times z) \% M$ which again contradicts the fact that $\beta$ is a maximum for the range up to $\beta$ since $\alpha-\omega<\alpha<\beta$.

Hence the result holds.
Second suppose that $((\alpha+\beta) \times z) \% M<(\beta \times z) \% M$. We want to show that $((\alpha+\beta) \times z) \% M$ is the new minimum while $(\beta \times z) \% M$ remains the maximum. Suppose that it is not the case, then at least one of the following two cases hold:

- If $((\alpha+\beta) \times z) \% M$ is not the new minimum, then there must be a new even smaller minimum. Thus there must be a value $\omega$ satisfying $0<\omega<\alpha$ such that $((\omega+\beta) \times z) \% M<((\alpha+\beta) \times z) \% M$. But this inequality implies that $(\omega \times z) \% M<(\alpha \times z) \% M$ which would contradict the fact that $(\alpha \times z) \% M$ was the minimum.
- If $(\beta \times z) \% M$ is no longer the maximum, then there must be a new larger maximum. Thus there must be a value $\omega$ satisfying $0<\omega<\alpha$ such that $((\omega+\beta) \times z) \% M>(\beta \times z) \% M$. Hence $((\omega+\beta) \times z) \% M=(\omega \times z) \% M+(\beta \times z) \% M$. Because $(\alpha \times z) \% M$ is the minimum up to $\beta$, we must have that $(\omega \times z) \% M>(\alpha \times z) \% M$. Therefore we have that $((\omega+\beta) \times z) \% M=(\omega \times$ z) $\% M+(\beta \times z) \% M>(\alpha \times z) \% M+(\beta \times z) \% M$. However, since $((\alpha+\beta) \times z) \% M<(\beta \times z) \% M$, we must have that $(\alpha \times z) \% M+(\beta \times z) \% M \geq M$ and so $((\omega+\beta) \times z) \% M \geq M$, a contradiction.

Hence the result holds.
Lemma 7.1 implies that you can visit all of the extrema of $(w \times z) \% M$ for $w=1,2, \ldots$ by first finding the first two extrema (a maximum at $\beta$ and a minimum at $\alpha$ ), and then locate a new maximum or a new
minimum at $\alpha+\beta$, and so forth. Through an iterative process, you are guaranteed to only ever visit running extrema.

Unfortunately, such a process could be slow. Consider the case when $z=1$. We have that the sequence $(w \times z) \% M$ for $w=1,2, \ldots$ is $1,2, \ldots$ It implies that every single possible value of $w$ is, when it is encountered, a new maximum. When applying Lemma 7.1 to this case, we find that $\alpha=1$ (throughout), with $\beta$ taking the values $2,3, \ldots$ Such an algorithm would encounter $M$ extrema and would run in time $\Omega(M)$. Thankfully, we can characterize the location of the extrema in logarithmic time, as we show next.

We have that $z \% M=(2 z) \% M$ if an only if $M$ divides $z$. Assume that $M$ does not divide $z$. As long as $w<M / \operatorname{gcd}(z, M)$, we have that $(w \times z) \% M \neq 0$.

Thus, after two elements in the series $(w \times z) \% M$ for $w=1,2, \ldots$, we have a minimum and a maximum value. We write the maximum value $b$, and its corresponding $w$ is $\beta$. We write the minimum value $a$, and its corresponding $w$ is $\alpha$. We have that $b>a>0$. As we keep progressing over $w \in[3, M / \operatorname{gcd}(M, z))$, we may encounter a new maximum or a new minimum.

Assume that the first value was a minimum (i.e., $a=z \% M$ and $\alpha=1$ ) followed by a maximum (i.e., $b=2 z \% M$ and $\beta=2$ ). If $3 a<M$, then we have a new maximum immediately after at $\beta=3$. Similarly if $4 a<M$ and so forth. We have exactly $(M-1-a) \div a$ consecutive maxima: $b=2 a$ at $\beta=2,3 a$ at $\beta=3, \ldots$, $a+((M-1-a) \div a) \times a$ at $\beta=1+(M-1-a) \div a$.

An analogous scenario unfolds when we assume that the first value was a maximum $(b=z \% M$ and $\beta=1)$ followed by a minimum $(a=2 z \% M$ and $\alpha=2)$. If $a+b \geq M$, then we have $(a+b) \% M=a+b-M<a+b$, and thus we have a new minimum (smaller by $M-b$ ). We have $b \div(M-b)$ consecutive minima: $b+b-M$ at $\alpha=2, b+2(M-b)$ at $\alpha=3, \ldots, b+(b \div(M-b)) \times(b-M)$ at $\beta=1+b \div(M-b)$. We have that the last maximum is greater than $M / 2$.

Using Lemma 7.1 and the first two extrema, we can efficiently iterate through all other extrema:

- Suppose that we found our last minimum at $\alpha$. We find a new maximum at $\beta(\alpha<\beta)$. By Lemma 7.1 this maximum is followed by up to $(M-b-1) \div a$ even greater maxima: $b+a$ at $w=\beta+\alpha, \ldots$, $b+((M-b-1) \div a) \times a$ at $w=\beta+((M-b-1) \div a) \times \alpha$. The maxima are each time incremented by $a$, and they appear at locations incremented by $\alpha$. If we were to continue one more step (increment by $a$ once more), we would exceed $M-1$. If we redefine $b \leftarrow b+((M-b-1) \div a) \times a$ and $\beta \leftarrow \beta+((M-b-1) \div a) \times \alpha$, then we have that the value at $\alpha+\beta$ is $(a+b) \% M=a+b-M<a$, thus a new minimum.
- Suppose that we found our last maximum at $\beta$. We find a new minimum at $\alpha(\beta<\alpha)$. By Lemma 7.1. this minima is followed by $a \div(M-b)$ even smaller minima: $a+b-M$ at $w=\alpha+\beta, \ldots$, $a+(a \div(M-b)) \times(b-M)$ at $w=\alpha+(a \div(M-b)) \times \beta$. They happen at locations separated by $\beta$ and decremented by $b-M$. If we were to continue one more step, we would increment by $b$ (as opposed to $b-M)$, and we would not have a new minimum. If we redefine $a \leftarrow a+(a \div(M-b)) \times(b-M)$ and $\alpha \leftarrow \alpha+(a \div(M-b)) \times \beta$, then we have that the value at $\alpha+\beta$ is $(a+b) \% M=a+b>b$, thus a new maximum.
Thus we have that $(w \times z) \% M$ for $w=$ $1,2, \ldots, M / \operatorname{gcd}(z, M)-1$ alternates between new equispaced sequences of maxima and new equispaced sequences of minima. We do not need to compute $\operatorname{gcd}(z, M)$ explicitly: we know that when $w=$ $M / \operatorname{gcd}(z, M)$, we have that $(w \times z) \% M=0$. Thus it suffices to check for a minimum value of 0 . The algorithm given in Fig. 2 outputs the sequence of gaps (successive $\alpha$ and $\beta$ ) which determine the locations of the extrema. Each gap value can generate several equispaced extrema.

We can check that with every iteration, going through a sequence of maxima, a sequence of minima and then back to a sequence of maxima, the gap $\beta$ has more than doubled. And similarly for $\alpha$. Thus we have that the algorithm given in Fig. 2 runs in time $O(\log (M / \operatorname{gcd}(z, M)))$ when assuming that arithmetic operations run in constant time.

Consider the algorithm of Fig. 2 with an example: $M=8$ and $z=3$. At first we have $a=b=(1 \times 3) \% 8=$ 3 and $w=\alpha=\beta=1$. The list lbda is initially empty.

1. We consider $v=(a+b) \% M=(3+3) \% 8=6$. It is a new maximum $(v>b)$. We compute $t=(M-b-1) \div a=(8-3-1) \div 3$ which is one. We append $w=1$ to lbda. We move to $w=1+1=2$ and set $\beta=2$. We have $b=6$ and $a=3$.
2. We consider $v=(a+b) \% M=(3+6) \% 8=1$. It is a new minimum $(v<a)$. We append $w=2$ to $\lambda$. We compute $a \div(M-b)=3 \div 2=1$. We move to $w=2+1=3$ and we set $\alpha=3$. We have $a=1$.
```
def \(\operatorname{gaps}(z, M):\)
    \(\mathrm{w}=1\)
    \(\mathrm{lbda}=[]\)
    \(\mathrm{a}=\mathrm{z} \% \mathrm{M}\)
    alpha \(=1\)
    \(\mathrm{b}=\mathrm{z} \% \mathrm{M}\)
    beta \(=1\)
    while True:
        \(\mathrm{v}=(\mathrm{a}+\mathrm{b}) \% \mathrm{M}\)
        if \(\mathrm{v}<\mathrm{a}\) :
            lbda.append(w)
            if a \(\%(\mathrm{M}-\mathrm{b})=0\) : break
            \(\mathrm{t}=\mathrm{a} / /(\mathrm{M}-\mathrm{b})\)
            \(\mathrm{w}=\mathrm{w}+\) alpha \(+(\mathrm{t}-1) *\) beta
            alpha \(=\mathrm{w}\)
            \(\mathrm{a}=(\mathrm{a}+\mathrm{t} * \mathrm{~b}) \% \mathrm{M}\)
        else:
            \(\mathrm{t}=(\mathrm{M}-\mathrm{b}-1) / / \mathrm{a}\)
            lbda.append(w)
            \(\mathrm{w}=\mathrm{w}+\operatorname{beta}+(\mathrm{t}-1) *\) alpha
            beta \(=\mathrm{w}\)
            \(b=(b+t * a) \% M\)
    return lbda
```

FIGURE 2: Python code to compute all of the gaps between the extrema of $(w \times z) \% M$ for $w=$ $1,2, \ldots, \frac{M}{\operatorname{gcd}(z, M)}-1 . \quad M$ and $z$ should be positive integers, and $z$ should not be a multiple of $M$.
3. We consider $v=(a+b) \% M=(1+6) \% 8=7$. We have a new maximum. We append $w=3$ to lbda. We compute $(M-b-1) \div a=(8-6-1) \div 1$ which is one, again. We move to $w=w+\beta=3+2=5$. We set $\beta=5$ and $b=7$.
4. We consider $(a+b) \% M=(1+7) \% 8=0$. It is a new minimum, we append $\beta=5$ to lbda and we exit the main loop, returning lbda $=\{1,2,3,5\}$.
If we compute $\operatorname{distance}_{8}(0,(w * 3) \% 8)$ for $w=$ $1,2,3,5$ we get $3,2,1,1$. That is, the distance of the various extrema to zero diminishes. It is clear that it must be so for successive (smaller) minima and also for successive (larger) maxima: the distance with zero must be strictly decreasing. Indeed a maximum is a value close to $M$, the closer to $M$, the larger it is. When we reach $M-1$, the largest value, we have that distance $_{M}(0, M-1)=1$, the minimal distance. But it is also true of a minimum followed by a maximum, or a maximum by a minimum: the distance with zero must remain the same or decrease. E.g., it follows by inspection: if we have a minimum value $a$, then it is not possible for the largest maximum of the next sequence of maxima to be more than $a$ away from $M$.

### 7.1. Bounding Remainders with an Offset

The algorithm of Fig. 2 provides an efficient algorithm to enumerate all the extrema of remainders $(w \times z) \% M$ for $w=1,2, \ldots$ We might want to enumerate the
extrema starting from an arbitrary point: $(w \times z) \% M$ for $w=A, A+1, \ldots, B$ in which case we can rewrite the problem as $(A \times z+w \times z) \% M$ for $w=0,1, \ldots, B-A$. Setting $b=A \times z$, we find that it is equivalent to finding the extrema of $(b+w \times z) \% M$ (for $w=0,1, \ldots)$. Thus we want to extend our previous results to remainder of a product with an offset $(b)$.

We cannot rely directly on the earlier result for $(w \times z) \% M$ (Lemma 7.1) which indicates that the next extrema is effectively the sum of the previous minima and the previous maxima. Indeed, consider $(7+w \times 3) \% 8$ : we have the value 2 at $w=1$, followed by value 5 (a new maximum) at $w=2$, value 0 (a new minimum) at $w=3$, value 6 (a new maximum) at $w=5$, value 7 (a new maximum) at $w=8$. However, we can still make good use of Lemma 7.1

Suppose that $(b+\beta \times z) \% M$ is the maximum so far over $(b+w \times z) \% M$ for $w=1,2, \ldots, \beta$. Suppose that the next extrema is at $(b+(\beta+k) \times z) \% M$. Then we must have that $(k \times z) \% M$ is a minimum of $(w \times z) \% M$ over $w=1,2, \ldots, k$ otherwise we would have a new intermediate extrema. And similarly when we start from a minima. It follows that we can access the extrema of $(b+w \times z) \% M$ by considering offsets by the gap values generated by the algorithm of Fig. 2. Because the gaps are monotonic, and because our maxima are only larger, and our minima only smaller, there is no need to consider previous gaps once they can no longer increase a maximum or decrease a minimum.

Even with a non-zero offset, the values still repeat: $(b+w \times z) \% M=(b+(w+M / \operatorname{gcd}(z, M)) \times z) \% M$. It is not necessary to compute $\operatorname{gcd}(z, M)$, we could instead stop when $w>M$ since no new extrema can be found after $w$ reaches $M / \operatorname{gcd}(z, M)$ given that the values repeat.

Thus the distance between the extrema of $(w \times$ z) $\% M$ for $w=A, A+1, \ldots, B$ must be within the values produced by the gaps function of Fig. 2, and that the gaps only grow larger. As an application, the find_min_max function from Fig. 3 computes the locations of all of the maxima and minima of $(w \times$ z) $\% b^{k}$ from $w=A$ to $w=B$ (inclusively). It encodes sequences of equispaced minima (or equispaced maxima) as a triple with the location of the last extrema, their number and the size of the gaps between the extrema. It runs in time $O(\log (M / \operatorname{gcd}(z, M)))$, assuming that arithmetic operations run in constant time.

## 8. COMPUTING THE RANGE

Our main function (find_range) is provided Fig. 4 given a multiplier $z$ and a desired number digits in a given base, it computes a range of values $[\mathrm{lb}, \mathrm{ub})$ such that if $w \in[\mathrm{lb}, \mathrm{ub})$, then $w \times z$ has its most significant digits exact even when $z$ is a truncated multiplierirrespective of the unknown digits. When the interval

```
def find_min_max \((z, M, A, B):\)
    mnma \(=\left[\begin{array}{lll}(A, & 0, & 0\end{array}\right]\)
    mxma \(=\left[\begin{array}{lll}(\mathrm{A}, & 0, & 0\end{array}\right]\)
    facts \(=\operatorname{gaps}(z, M)\)
    \(m i=(z * A) \% M\)
    \(m a=(z * A) \% M\)
    fact_index \(=0\)
    \(\mathrm{b}=\mathrm{A} * \mathrm{z}\)
    \(\mathrm{w}=0\)
    while True:
        offindex \(=\) facts [fact_index]
        \(\mathrm{v}=(\mathrm{z} *(\mathrm{w}+\) offindex \()+\mathrm{b}) \% \mathrm{M}\)
        if \(w+\) offindex \(>B-A:\) break
        if \(\mathrm{v}<\mathrm{mi}\) :
            \(\mathrm{w}+=\) offindex
            \(\mathrm{mi}=\mathrm{v}\)
            basis \(=(\mathrm{z} * \mathrm{w}+\mathrm{b}) \% \mathrm{M}\)
            off \(=(z *\) offindex \() \% M\)
            times \(=\) basis \(/ /(M-\) off \()\)
            if \(\mathrm{A}+\mathrm{w}+\) times \(*\) offindex \(>\mathrm{B}\) :
            times \(=(B-A-w) / /\) offindex
            \(\mathrm{w}+=\) offindex \(*\) times
            \(\mathrm{mi}=(\mathrm{z} * \mathrm{w}+\mathrm{b}) \% \mathrm{M}\)
            mnma. append ( \((\mathrm{w}+\mathrm{A}\), times, offindex \())\)
        elif \(v>m\) m:
            \(\mathrm{w}+=\) offindex
            \(\mathrm{ma}=\mathrm{v}\)
            basis \(=(\mathrm{z} * \mathrm{w}+\mathrm{b}) \% \mathrm{M}\)
            off \(=(\mathrm{z} *\) offindex) \(\% \mathrm{M}\)
            times \(=(M-1-\) basis \() / /\) off
            if \(\mathrm{A}+\mathrm{w}+\) times \(*\) offindex \(>\mathrm{B}\) :
            times \(=(B-A-w) / /\) offindex
            \(\mathrm{w}+=\) offindex \(*\) times
            \(\mathrm{ma}=(\mathrm{z} * \mathrm{w}+\mathrm{b}) \% \mathrm{M}\)
            mxma. append ( \((w+A\), times, offindex \())\)
        else:
            fact_index \(+=1\)
            if fact_index \(=\) len (facts): break
    return (mnma, mxma)
```

FIGURE 3: Python code to enumerate all extrema of $(w \times z) \% M$ for $w=A, \ldots, B . M$ and $z$ should be positive integers, and $z$ should not be a multiple of $M$.
is empty, the None value is returned.
Following § 6, the function checks $(w \times z) \% b^{k}<$ $b^{k}-w+1$ over the range $b^{d+k-1} \leq(w \times z) \leq b^{d+k}-1$ to verify whether we can compute digits exactly: starting with $k=0$, we increment $k$ until a value $w$ satisfying $(w \times z) \% b^{k} \geq b^{k}-w+1$ is found. To do so, the find_range function relies on the find_min_max function from Fig. 3. We handle separately the case when $z$ is a multiple of $b^{k}$, in which case $(w \times z) \% b^{k}=0$ and $(w \times z) \% b^{k}<b^{k}-w+1$ becomes $w<b^{k}+1$ : thus we must stop if $B \geq b^{k}+1$.

We can generate Table 1 with a Python script:

```
for mypi in [
    3141592653,
    31415926535,
```

```
def find_range(z, digits, base):
    \(\mathrm{lb}=(\) base \(* *(\operatorname{digits}-1)+\mathrm{z}-1) / / \mathrm{z}\)
    \(\mathrm{k}=0\)
    while True:
    \(\mathrm{A}=(\) base \(* *(\) digits \(+\mathrm{k}-1)+\mathrm{z}-1) / / \mathrm{z}\)
    \(\mathrm{B}=\) base \(* *(\) digits +k\() / / \mathrm{z}\)
    if \(\mathrm{B} * \mathrm{z}=\) base \(* *\) (digits +k ):
        B \(-=1\)
    if \(\mathrm{B}<\mathrm{A}\) :
        \(\mathrm{k}=\mathrm{k}+1\)
        continue
    \(\mathrm{M}=\) base \(* * \mathrm{k}\)
    if \(\mathrm{z} \% \mathrm{M}=0\) : \# special case
            if \(\mathrm{B}>=\mathrm{M}+1\) :
                return ( \(\mathrm{lb}, \mathrm{M}+1\) )
        (mnma, mxma) \(=\) find_min_max \((z, M, A, B)\)
        for (beta, times, gap) in mxma:
        if beta * \(\mathrm{z} \% \mathrm{M}>=\mathrm{M}-\mathrm{beta}+1\) :
            \(\mathrm{ub}=\) beta
            if times \(>0\) :
                top \(=\) beta \(*\) z \(\% \mathrm{M}-(\mathrm{M}-\) beta +1\()\)
                bottom \(=\) gap + gap \(* \mathrm{z} \% \mathrm{M}\)
            \(\mathrm{mt}=\) top // bottom
            \(\mathrm{ub}=\) beta - top // bottom * gap
        if \(\mathrm{ub}<=\mathrm{lb}\) :
            return None
        return (lb, ub)
    \(\mathrm{k}=\mathrm{k}+1\)
```

FIGURE 4: Python code to compute the range of values of $w$ for which the most significant digits of $w \times z$ are exact even if $z$ is truncated. The function allows the user to specify the number of exact most-significant digits desired (digits) as well as the basis (base).

```
314159265358,
3141592653589,
31415926535897,
314159265358979,
3141592653589793,
31415926535897932,
314159265358979323,
3141592653589793238,
31415926535897932384,
]:
print(find_range(mypi, 10, 10))
```


## 9. CONCLUSION AND FUTURE WORK

It is intuitive that we are able to compute the most significant digits of a product using only the most significant digits of one of the multipliers (i.e., a short multiplier). Thankfully, we can check for the exact range of validity of a short multiplier using efficient logarithmic-time algorithms.

We have identified some future work:

- We construct short multipliers by truncating existing multipliers. However it may be possible to round the multiplier instead of truncating it.

Similarly, we compute the most significant digits of a product by truncation: we may round instead. There are many rounding rules that should be considered when the result is ambiguous: round up, round down, round to even [6], round to odd.

- From a short multiplier and a desired number of most-significant digits, we have derived a range of validity. We can also start by a desired number of most-significant digits and a range of validity, and derive the smallest short multiplier: it suffices to construct ever more precise short multipliers. A direct algorithm to efficiently derive the best short multipliers might be useful.


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## 11. DATA AVAILABILITY

No new data were generated or analysed in support of this research.

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[^0]:    ${ }^{1}$ https://github.com/lemire/exactshortlib

